Structures of Diversity

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ICTCS 2010
**Zero–Error Capacity**

Shannon 1956*

Suppose we want to transmit messages across a channel (where some symbols may be distorted) to a receiver: What is the maximum rate of transmission such that the receiver may recover the original message without errors?

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<table>
<thead>
<tr>
<th>Channel Alphabet</th>
<th>V</th>
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<tbody>
<tr>
<td></td>
<td>1, 2, 3, 4, 5</td>
</tr>
</tbody>
</table>

Distinguishable symbols

Transmission Rate: The maximum number of bits that can be transmitted without errors per channel use.

Suppose we want to transmit messages across a channel (where some symbols may be distorted) to a receiver: What is the maximum rate of transmission such that the receiver may recover the original message without errors?

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   Single symbols: \( \log_2 \)

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Suppose we want to transmit messages across a channel (where some symbols may be distorted) to a receiver: What is the maximum rate of transmission such that the receiver may recover the original message without errors?

**Channel**

Alphabet $V = \{1, 2, 3, 4, 5\}$

**Transmission Rate:** The maximum number of bits that can be transmitted without errors per channel use.

Single symbols: $\log_2(5)$

Zero-error capacity

... and if we use larger strings in place of single symbols ...
Zero-error capacity

... and if we use larger strings in place of single symbols ...

\[ x_1 x_2 \in y_1 y_2 \]
Zero-error capacity

... and if we use larger strings in place of single symbols ...

\[ x_1 x_2 \in y_1 y_2 \]
Zero-error capacity

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\( x_1 x_2 \oplus y_1 y_2 \)
Zero-error capacity

... and if we use larger strings in place of single symbols ...

Graph $G^2$: 
Zero-error capacity

... and if we use larger strings in place of single symbols ...

Graph $G^2$:

$V(G^2) = V \times V = \{11, 12, \ldots, 55\}$
Zero-error capacity

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Graph $G^2$:

- $V(G^2) = V \times V = \{11, 12, \ldots, 55\}$
- $\{v, w\} \in E(G^2) \Rightarrow \exists i : \{v_i, w_i\} \in E(G)$
Zero-error capacity

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$\omega(G^2) = ?$
Zero-error capacity

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$C = \{11, 23, 35, 42, 54\}$
Zero-error capacity

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Transmission Rate: $\frac{1}{2} \log 5 > \log 2$
Zero-error capacity

... and if we use larger strings in place of single symbols ...

Graph $G^2$:
- $V(G^2) = V \times V = \{11, 12, \ldots, 55\}$
- $\{v, w\} \in E(G^2) \Rightarrow \exists i : \{v_i, w_i\} \in E(G)$
  
\[ \omega(G^2) = ? \]

$C = \{11, 23, 35, 42, 54\}$

Transmission Rate: $\frac{1}{2} \log 5 > \log 2$

If $C$ is a clique in $G^n$ then

\[ x, y \in C \Rightarrow \exists i \in [n], \{x_i, y_i\} \in E(G). \]
Definition

The Shannon **zero-error capacity** of $G$ is

$$C(G) = \lim_{n \to +\infty} \frac{1}{n} \log \omega(G^n)$$
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$C(C_5) = ?$
**Definition**

The Shannon zero-error capacity of $G$ is

$$C(G) = \lim_{n \to +\infty} \frac{1}{n} \log \omega(G^n)$$

$C(C_5) = \frac{1}{2} \log 5$

Lovász ’79: $C(C_5) = \frac{1}{2} \log 5$

The Shannon zero-error capacity of $G$ is

$$C(G) = \lim_{n \to +\infty} \frac{1}{n} \log \omega(G^n)$$

$\mathbf{C}(C_5) = {?}$

Lovász ’79: $C(C_5) = \frac{1}{2} \log 5$

Determining the value of $C(C_7)$ is still open!

Generalizations

- Graphs [Sha56]
- Directed Graphs [KS92, GKV92]
- Graph Families [CKS90, GKV94]
- Uniform Hypergraphs [KM90]
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Connections

Extremal Combinatorics

- Perfect Graphs [Ber62]
- Qualitative Independence [Rén71, GKV93]

Information Theory

- Perfect hashing [FK84]
- Zero error list decoding [Eli57]
- Zero error capacity of compound channels [BBT59, Dob59, Wol60, NR05]
Generalization to *Infinite Graphs*?
Generalization to *Infinite Graphs*?
Generalization to *Infinite Graphs*?

↓

*Permutations*
**G–different Permutations.**

A precise Result
Shannon Zero–Error Capacity
Difference and Similarity

\[ \pi = \pi(1)\pi(2) \ldots \pi(n) \]
Definition

$G$ an infinite graph with $V(G) = \mathbb{N}$. Two permutations $\pi, \rho$ of $[n]$ are said $G$–different if $\exists i \in [n]$ such that $\{\pi(i), \rho(i)\} \in E(G)$. 

Example

$G$: $n = 5$
$\pi = 12345$
$\rho = 13245$
$G$–different.

$\pi = 12345$
$\rho = 34125$
Not $G$–different.
Definition

$G$ an infinite graph with $V(G) = \mathbb{N}$. Two permutations $\pi, \rho$ of $[n]$ are said $G$–different if $\exists i \in [n]$ such that $\{\pi(i), \rho(i)\} \in E(G)$.

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**Example**

$$G : \begin{array}{c}
1 & 2 & 3 & 4 \\
\bullet & \bullet & \bullet & \bullet
\end{array}$$
Definition

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$G : \begin{array}{cccc}
1 & 2 & 3 & 4 \\
\cdot & \cdot & \cdot & \cdot \\
\end{array}$

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Example

$G$:

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & \cdots \\
\end{array}
\]

$n = 5$

\begin{itemize}
  \item $\pi = 12345$
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### Problem

\( T(G, n) \) the maximum cardinality of a set of pairwise \( G \)-different permutations of \([n]\).
Problem

$T(G, n)$ the maximum cardinality of a set of pairwise $G$–different permutations of $[n]$. 

$T(G, n) = ?$
Problem

\( T(G, n) \) the maximum cardinality of a set of pairwise \( G \)-different permutations of \([n]\).

\[ T(G, n) = ? \]

The semi–infinite path \( L \)

Conjecture

[KM06]

\[ T(L, n) = \binom{n}{\left\lfloor \frac{n}{2} \right\rfloor} \]
### Problem

\( T(G, n) \) the maximum cardinality of a set of pairwise \( G \)-different permutations of \([n]\).

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The semi-infinite path \( L \)

### Conjecture

\[ [KM06] \]

\[
T(L, n) = \binom{n}{\lfloor \frac{n}{2} \rfloor}
\]

- Best upper and lower bounds in \[BCF^+\]
Problem

\( T(G, n) \) the maximum cardinality of a set of pairwise 
\( G \)-different permutations of \([n]\).

\[ T(G, n) = ? \]

The semi–infinite path \( L \)

Conjecture

[KM06]

\[ T(L, n) = \binom{n}{\lfloor n/2 \rfloor} \]

- Best upper and lower bounds in [BCF+]

Surprisingly for the complement graph of \( L \) an exact formula is found ...
Let $\overline{L}$ be the complement graph of semi-infinite path, then

$$T(n, \overline{L}) = \frac{n!}{2\left\lfloor \frac{n}{2} \right\rfloor}$$

**Theorem**
Theorem

Let $\bar{L}$ be the complement graph of semi-infinite path, then

$$T(n, \bar{L}) = \frac{n!}{2^{\lceil \frac{n}{2} \rceil}}$$

Proof ($\leq$)
Theorem

Let \( \overline{L} \) be the complement graph of semi-infinite path, then

\[
T(n, \overline{L}) = \frac{n!}{2^\left\lfloor \frac{n}{2} \right\rfloor}
\]

Proof (\( \leq \))
For any \( \pi \) a permutation of \([n]\) define a set \( C(\pi) \):
Theorem

Let $\bar{L}$ be the complement graph of semi-infinite path, then

$$T(n, \bar{L}) = \frac{n!}{2^{\left\lfloor \frac{n}{2} \right\rfloor}}$$

Proof ($\leq$)

For any $\pi$ a permutation of $[n]$ define a set $C(\pi)$:

$$\pi : 231456$$
**Theorem**

Let \( \overline{L} \) be the complement graph of semi-infinite path, then

\[
T(n, \overline{L}) = \frac{n!}{2\lceil \frac{n}{2} \rceil}
\]

**Proof** \((\leq)\)

For any \( \pi \) a permutation of \([n]\) define a set \( C(\pi) \):

\[\pi : 231456\]
Theorem

Let $\bar{L}$ be the complement graph of semi-infinite path, then

$$T(n, \bar{L}) = \frac{n!}{2^\left\lfloor \frac{n}{2} \right\rfloor}$$

Proof ($\leq$)

For any $\pi$ a permutation of $[n]$ define a set $C(\pi)$:

\[
\pi : 231456 \\
1 \leftrightarrow 2 : 132456
\]
Let $\overline{L}$ be the complement graph of semi-infinite path, then

$$T(n, \overline{L}) = \frac{n!}{2^\left\lfloor \frac{n}{2} \right\rfloor}$$

**Proof** ($\leq$)

For any $\pi$ a permutation of $[n]$ define a set $C(\pi)$:

$$\pi : \ {231456}$$

$$1 \leftrightarrow 2 : \ {132456}$$

$$3 \leftrightarrow 4 : \ {241356}$$
**Theorem**

Let $\overline{L}$ be the complement graph of semi-infinite path, then

$$T(n, \overline{L}) = \frac{n!}{2^\lfloor \frac{n}{2} \rfloor}$$

**Proof** ($\leq$)

For any $\pi$ a permutation of $[n]$ define a set $C(\pi)$:

- $\pi : 231456$
- $1 \leftrightarrow 2 : 132456$
- $3 \leftrightarrow 4 : 241356$
- $1 \leftrightarrow 2, 5 \leftrightarrow 6 : 132465$
**Theorem**

Let \( \bar{L} \) be the complement graph of semi-infinite path, then

\[
T(n, \bar{L}) = \frac{n!}{2^\left\lfloor \frac{n}{2} \right\rfloor}
\]

**Proof** \((\leq)\)

For any \( \pi \) a permutation of \([n]\) define a set \( C(\pi) \):

- \( \pi : 231456 \)
- \( 1 \leftrightarrow 2 : 132456 \)
- \( 3 \leftrightarrow 4 : 241356 \)
- \( 1 \leftrightarrow 2, 5 \leftrightarrow 6 : 132465 \)

- \( |C(\pi)| = 2^\left\lfloor \frac{n}{2} \right\rfloor \)
- If \( \pi, \rho \) are \( \bar{L} \)-different then \( C(\pi) \cap C(\rho) = \emptyset \).
Proof (≥)

\[ T(n - 1, D) \geq T(n - 2, L) \]
**Proof (≥)**

**IDEA:**
**Proof ($\geq$)**

**IDEA:**

\[
\begin{align*}
T(n - 2, D) \geq T(n - 1, D)
\end{align*}
\]
**Proof** $(\geq)$

**IDEA:**

$$T(n-2, D)$$

Insert $n - 1$ and $n$
Proof \((\geq)\)

**IDEA:**

\[
T(n - 2, D) \geq \binom{n}{2} T(n - 2, \bar{L})
\]
Asymptotic Growth $T(n,D)$
**Asymptotic Growth** \( T(n,D) \)

- Exponential \( \sim \exp(n) \):
**Asymptotic Growth** $T(n, D)$

- Exponential $\sim \exp(n)$:
  \[
  1.8155^n \leq T(n, L) \leq \left(\frac{n}{\lfloor n/2 \rfloor}\right)
  \]
Asymptotic Growth $T(n,D)$

- Exponential $\sim \exp(n)$: $1.8155^n \leq T(n, L) \leq \left( \frac{n}{\lfloor n/2 \rfloor} \right)$

- Super-Exponential $\sim \frac{n!}{\exp(n)}$:
**Asymptotic Growth** \( T(n,D) \)

- **Exponential** \( \sim \exp(n) \): \( 1.8155^n \leq T(n,L) \leq \binom{n}{\lfloor n/2 \rfloor} \)

- **Super-Exponential** \( \sim \frac{n!}{\exp(n)} \): \( T(n,\bar{L}) = \frac{n!}{2^{\lfloor n/2 \rfloor}} \)
Asymptotic Growth $T(n,D)$

- Exponential $\sim \exp(n)$: $1.8155^n \leq T(n, L) \leq {n \choose \lfloor n/2 \rfloor}$

- Super-Exponential $\sim \frac{n!}{\exp(n)}$: $T(n, \overline{L}) = \frac{n!}{2^{\lfloor n/2 \rfloor}}$

- Other?
**Asymptotic Growth** \( T(n, D) \)

- **Exponential** \( \sim \exp(n) \):
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  1.8155^n \leq T(n, L) \leq \left( \frac{n}{\lfloor \frac{n}{2} \rfloor} \right)
  \]

- **Super-Exponential** \( \sim \frac{n!}{\exp(n)} \):
  \[
  T(n, \overline{L}) = \frac{n!}{2^{\lfloor \frac{n}{2} \rfloor}}
  \]

- **Other?**
  \[
  (\sqrt{n})!^{\sqrt{n}} \leq T(n, F) \leq \frac{n!}{(\sqrt{n})!^{\sqrt{n}}}
  \]
**Asymptotic Growth** $T(n, D)$

- Exponential $\sim \exp(n)$: 
  \[1.8155^n \leq T(n, L) \leq \left(\frac{n}{\lfloor n/2 \rfloor}\right)\]

- Super-Exponential $\sim \frac{n!}{\exp(n)}$: 
  \[T(n, \bar{L}) = \frac{n!}{2^{\lfloor n/2 \rfloor}}\]

- Other? 
  \[\sqrt{n}! \sqrt[n]{n} \leq T(n, F) \leq \frac{n!}{(\sqrt{n})! \sqrt[n]{n}}\]

*Study the relations between* $T(n, G)$ *and* $T(n, \bar{G})$
**Asymptotic Growth** $T(n,D)$

- Exponential $\sim \exp(n)$: $1.8155^n \leq T(n,L) \leq \left( \frac{n}{\lfloor \frac{n}{2} \rfloor} \right)$

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  $$(\sqrt{n})!^{\sqrt{n}} \leq T(n,F) \leq \frac{n!}{(\sqrt{n})!^{\sqrt{n}}}$$

**Study the relations between** $T(n,G)$ **and** $T(n,\overline{G})$

**Study the asymptotic of**

$$\frac{T(n,F)T(n,G)}{T(n,F \cup G)}$$
The Shannon zero–error capacity is a special case of the problem of determining the asymptotic growth of $T(n, G)$.
Example

Consider $G$ with $V(G) = \mathbb{N}$ and $\{a, b\} \in E(G)$ if $|a - b| \equiv 1 \circ 4 \pmod{5}$. 
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Consider $G$ with $V(G) = \mathbb{N}$ and $\{a, b\} \in E(G)$ if $|a - b| \equiv 1 \circ 4 \pmod{5}$. 

![Graph with nodes 0, 1, 2, 3, 4 connected in a cycle]
Consider $G$ with $V(G) = \mathbb{N}$ and $\{a, b\} \in E(G)$ if $|a - b| \equiv 1 \circ 4 \pmod{5}$.

\[
\lim_{n \to +\infty} \frac{1}{n} \log T(n, G) = C(C_5)
\]
Difference and Similarity

G–difference
Difference and Similarity

**G–difference**

- Irreflexive relation
Difference and Similarity

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- Locally verifiable
Difference and Similarity

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The “opposite” of a difference relation
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The “opposite” of a difference relation

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**Intersection Problems**
Difference and Similarity

G–difference

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The “opposite” of a difference relation

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Intersection Problems

- Erdős–Ko–Rado [EKR61]
Difference and Similarity

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Intersection Problems
- Erdős–Ko–Rado [EKR61]
- Intersection theorems for permutations [EFP]
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**Intersection Problems**
- Erdős–Ko–Rado [EKR61]
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Similarity relation

The $G$–difference property is never satisfied!!
Difference and Similarity

**G–difference**
- Irreflexive relation
- Locally verifiable

The “opposite” of a difference relation

**Similarity relation**
- Reflexive relation
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**Intersection Problems**
- Erdős–Ko–Rado [EKR61]
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Similarity relation

The $G$–difference property is never satisfied!!

- Reflexive relation
Difference and Similarity

**G–difference**
- Irreflexive relation
- Locally verifiable

The "opposite" of a difference relation

**Similarity relation**
- Reflexive relation
- Locally verifiable

**Intersection Problems**

**Similarity relation**

The $G$–difference property is never satisfied!!

- Reflexive relation
- Not locally verifiable

**Intersection theorems for permutations [EFP]**

**Erdős–Ko–Rado [EKR61]**
Forbiddance Problems

Definition

For $G$, denote by $\tilde{G}(G)$ the family of all the orientations of $G$. 
For $G$, denote by $\vec{G}(G)$ the family of all the orientations of $G$.

A **capacity** type problem

Find the maximum cardinality of $C \subseteq [V(G)]^n$ such that $x, y \in C$ and for any $G' \in \vec{G}(G)$ $\exists i, j$ for which $(x_i, y_i)$ and $(y_j, x_j)$ are in $E(G')$. 
Definition

For $G$, denote by $\tilde{G}(G)$ the family of all the orientations of $G$.

A capacity type problem

Find the maximum cardinality of $C \subseteq [V(G)]^n$ such that $x, y \in C$ and for any $G' \in \tilde{G}(G) \exists i, j$ for which $(x_i, y_i)$ and $(y_j, x_j)$ are in $E(G')$.
Definition

For $G$, denote by $\vec{G}(G)$ the family of all the orientations of $G$.

A capacity type problem

Find the maximum cardinality of $C \subseteq [V(G)]^n$ such that $x, y \in C$ and for any $G' \in \vec{G}(G)$ $\exists i, j$ for which $(x_i, y_i)$ and $(y_j, x_j)$ are in $E(G')$.
- $G = K_N$
- permutations
\begin{itemize}
  \item $G = K_N$
  \item permutations
\end{itemize}

**Capacity**

**Reverse–different**
\[ G = K_N \]

permutations

**Capacity**

**Reverse–different**

\[
\begin{array}{cccccc}
1 & 2 & 5 & 4 & 3 & 6 \\
1 & 3 & 4 & 6 & 2 & 5 \\
\end{array}
\]
\[ G = K_N \]

\[ \text{permutations} \]

**Capacity**

**Reverse–different**

\[
\begin{array}{cccccc}
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\end{array}
\]
Forbiddance of a graph family
Reverse–free triples

Background
G–different permutations
Forbiddance Problems
2–Cancellative Families
Conclusion

- $G = K_N$
- permutations

Capacity
Reverse–different

Forbiddance
Reverse–Free

$1 \ 2 \ 5 \ 4 \ 3 \ 6$

$1 \ 3 \ 4 \ 6 \ 2 \ 5$
**Forbiddance of a graph family**

**Reverse–free triples**

- $G = K_N$
- permutations

### Capacity

**Reverse–different**

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<th>1</th>
<th>2</th>
<th>5</th>
<th>4</th>
<th>3</th>
<th>6</th>
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### Forbiddance

**Reverse–Free**

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**Reverse–different**

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\( G = \mathcal{K}_N \)

**Capacity**

**Reverse-different**

\[
\begin{array}{cccccc}
1 & 2 & 5 & 4 & 3 & 6 \\
\hline
1 & 3 & 4 & 6 & 2 & 5
\end{array}
\]

**Forbiddance**

**Reverse-Free**

\[
\begin{array}{cccccc}
1 & 2 & 5 & 4 & 3 & 6 \\
\hline
1 & 5 & 4 & 3 & 6 & 2
\end{array}
\]

\[
2 \left\lfloor \frac{n}{2} \right\rfloor \leq T(n) \leq \frac{n!}{3 \left\lfloor \frac{n}{2} \right\rfloor}
\]
G = $K_N$

Forbiddance

Reverse–different

\[ 2 \lfloor \frac{n}{2} \rfloor \leq T(n) \leq \frac{n!}{3 \lfloor \frac{n}{2} \rfloor} \]

Reverse–Free

\[ 3 \lfloor \frac{n}{2} \rfloor \leq T'(n) \leq \frac{n!}{2 \lfloor \frac{n}{2} \rfloor} \]
Reverse–free permutations
Reverse–free permutations

Partial permutations, i.e. ordered sets of $k$-elements.
Reverse–Free $k$-uples

Let $T_k(n)$ be the maximum cardinality of a set of reverse-free $k$-uples of $[n]$. 
Let $T_k(n)$ be the maximum cardinality of a set of reverse-free $k$-uples of $[n]$.

$$t(k) = \limsup_{n \to \infty} \frac{T_k(n)}{k! \binom{n}{k}} = ?$$
Reverse–Free $k$-uples

Let $T_k(n)$ be the maximum cardinality of a set of reverse-free $k$-uples of $[n]$.

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If $k = 2$ then $T_2(n) = \binom{n}{2} \Rightarrow t(2) = \frac{1}{2}$
Reverse–Free $k$-uples

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If $k = 3$ ??
Reverse–Free $k$-uples

Let $T_k(n)$ be the maximum cardinality of a set of reverse-free $k$-uples of $[n]$. 

$$t(k) = \limsup_{n \to \infty} \frac{T_k(n)}{k!(\binom{n}{k})} = ?$$

If $k = 2$ then $T_2(n) = \binom{n}{2} \Rightarrow t(2) = \frac{1}{2}$

If $k = 3$ ?? $t(3) = 5/24$
Case $k = 3$
Case $k = 3$

Theorem

$$T_3(n) = \frac{5}{24} n^3 - \frac{1}{2} n^2 + \frac{5}{8} n$$
Case $k = 3$

**Theorem**

$$T_3(n) = \frac{5}{24}n^3 - \frac{1}{2}n^2 + \frac{5}{8}n$$

**Proof**
Case $k = 3$

**Theorem**

$$T_3(n) = \frac{5}{24} n^3 - \frac{1}{2} n^2 + \frac{5}{8} n$$

**Proof**

($\geq$) Recursive construction. Tight when $n = 3^q$.
Case \( k = 3 \)

**Theorem**

\[
T_3(n) = \frac{5}{24} n^3 - \frac{1}{2} n^2 + \frac{5}{8} n
\]

**Proof**

(\( \geq \)) Recursive construction. Tight when \( n = 3^q \).

(\( \leq \))
Case $k = 3$

Theorem

$$T_3(n) = \frac{5}{24}n^3 - \frac{1}{2}n^2 + \frac{5}{8}n$$

Proof

(≥) Recursive construction. Tight when $n = 3^q$.

(≤) For any $a, b, c \in [n]$
Case $k = 3$

**Theorem**

$$T_3(n) = \frac{5}{24} n^3 - \frac{1}{2} n^2 + \frac{5}{8} n$$

**Proof**

$(\geq)$ Recursive construction. Tight when $n = 3^q$.

$(\leq)$ For any $a, b, c \in [n]$

```
  a  b  c
  b  c  a
  c  a  b
```

```
  b  a  c
  a  c  b
  c  b  a
```
Case $k = 3$

**Theorem**

$$T_3(n) = \frac{5}{24} n^3 - \frac{1}{2} n^2 + \frac{5}{8} n$$

**Proof**

($\geq$) Recursive construction. Tight when $n = 3^q$.

($\leq$) For any $a, b, c \in [n]$

\[
\begin{align*}
\text{a} & \quad \text{b} & \quad \text{c} \\
\text{b} & \quad \text{c} & \quad \text{a} \\
\text{c} & \quad \text{a} & \quad \text{b} \\
\text{b} & \quad \text{a} & \quad \text{c} \\
\text{a} & \quad \text{c} & \quad \text{b} \\
\text{c} & \quad \text{b} & \quad \text{a}
\end{align*}
\]
Case $k = 3$

**Theorem**

$$T_3(n) = \frac{5}{24} n^3 - \frac{1}{2} n^2 + \frac{5}{8} n$$

**Proof**

(≥) Recursive construction. Tight when $n = 3^q$.

(≤) For any $a, b, c \in [n]$:

\[
\begin{align*}
& a & b & c \\
& b & c & a \\
& c & a & b \\
& b & a & c \\
& a & c & b \\
& c & b & a \\
\end{align*}
\]
IDEA

\[ abc \in C \]
$abc \in C$
IDEA

$abc \in C$
\[ abc \in C \]
IDEA

$abc \in C$
Forbiddance of a graph family
Reverse–free triples

IDEA

G – different permutations
Forbiddance Problems
2–Cancellative Families
Conclusion

D with E (D) = E (G1) ∩ E (G2) ∩ E (G3)

\[
\begin{align*}
G_1 & : a \rightarrow b, c, e, d \\
G_2 & : a \rightarrow b, c, e, d \\
G_3 & : a \rightarrow b, c, e, d
\end{align*}
\]
D with $E(D) = E(G_1) \cap E(G_2) \cap E(G_3)$
IDEA

\[
D \text{ with } E(D) = E(G_1) \cap E(G_2) \cap E(G_3)
\]
Cancellative Families

Capacity of a $k$-uniform hypergraph

$G$–difference: from binary relations to relations involving $k$-sets of strings
Capacity of a \( k \)-uniform hypergraph

\( G \)-difference: from binary relations to relations involving \( k \)-sets of strings

\( \downarrow \)
G–different permutations

Forbiddance Problems

2–Cancellative Families

Conclusion

Capacity of a uniform hypergraph

Requirements over three strings

Requirements over four strings

Capacity of a $k$-uniform hypergraph

G–difference: from binary relations to relations involving $k$-sets of strings

\[ \downarrow \]

Capacity of a $k$-uniform hypergraph
Capacity of a $k$-uniform hypergraph

$G$–difference: from binary relations to relations involving $k$-sets of strings

$\downarrow$

Capacity of a $k$-uniform hypergraph
\( H \) is a complete \textbf{k–uniform} hypergraph
$H$ is a complete $k$–uniform hypergraph

Problem [KS88]
Let $C \subseteq [V]^n$ such that for any $k$ strings $\exists I_m, |V| \subseteq [n]$ where the projections of the strings are all different.
$H$ is a complete $k$–uniform hypergraph

Problem [KS88]
Let $C \subseteq [V]^n$ such that for any $k$ strings \( \exists I_m, |V| \subseteq [n] \) where the projections of the strings are all different.

First case to consider: $V = \{0, 1\}$ e $k = 4$
$H$ is a complete $k$–uniform hypergraph

**Problem [KS88]**

Let $C \subseteq [V]^n$ such that for any $k$ strings $\exists I_{m,|V|} \subseteq [n]$ where the projections of the strings are all different.

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First case to consider: $V = \{0, 1\}$ and $k = 4$
Problem [KS88]

Let $C \subseteq [V]^n$ such that for any $k$ strings $\exists I_{m,|V|} \subseteq [n]$ where the projections of the strings are all different.

First case to consider: $V = \{0, 1\} \; e \; k = 4$

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Variations:
$H$ is a complete $k$–uniform hypergraph

**Problem [KS88]**

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First case to consider: $V = \{0, 1\}$ and $k = 4$

Variations:

- One column of weight two
$H$ is a complete $k$–uniform hypergraph

**Problem [KS88]**

Let $C \subseteq [V]^n$ such that for any $k$ strings $\exists I_m, |V| \subseteq [n]$ where the projections of the strings are all different.

First case to consider: $V = \{0, 1\}$ and $k = 4$

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Variations:

- One column of weight two (two columns?)
A complete $k$–uniform hypergraph

**Problem [KS88]**

Let $C \subseteq [V]^n$ such that for any $k$ strings $\exists I_m, |V| \subseteq [n]$ where the projections of the strings are all different.

\[
\begin{array}{cc}
i & j \\
0 & 0 \\
0 & 1 \\
1 & 0 \\
1 & 1 \\
\end{array}
\]

First case to consider : $V = \{0, 1\}$ e $k = 4$

Variations:

- One column of weight two (two columns?)
- One column of weight one
$H$ is a complete $k$–uniform hypergraph

**Problem [KS88]**

Let $C \subseteq [V]^n$ such that for any $k$ strings $\exists I_m, |V| \subseteq [n]$ where the projections of the strings are all different.

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First case to consider : $V = \{0, 1\}$ and $k = 4$

Variations:

- One column of weight two (two columns?)
- One column of weight one (two?, three? …)
$H$ is a complete $k$–uniform hypergraph

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Variations:

- One column of weight two (two columns?)
- One column of weight one (two?, three? ...)
- $k = 3$?
\( H \) is a complete \textbf{k-uniform} hypergraph

\textbf{Problem [KS88]}

Let \( C \subseteq [V]^n \) such that for any \( k \) strings \( \exists \{m, |V| \subseteq [n] \) where the projections of the strings are all different.

\[
\begin{array}{c|c|c}
0 & 0 & 1 \\
1 & 0 & 1 \\
1 & 1 & 1 \\
\end{array}
\]

First case to consider: \( V = \{0, 1\} \) e \( k = 4 \)

Variations:

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- One column of weight one (two?, three? ...)
- \( k = 3 \) ?

\textbf{Open Problems!!}
Requirements over three strings

**Problem**

Determine the maximum cardinality of $C \subseteq [V]^n$ such that for any 3 of its elements there exists $r$ coordinates in which the respective columns of the strings are all different and of weight 1.
Problem

Determine the maximum cardinality of $C \subseteq [V]^n$ such that for any 3 of its elements there exists $r$ coordinates in which the respective columns of the strings are all different and of weight 1.

- $r = 1$ : $\Delta$–systems [ES78, Kos00]
Requirements over three strings

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- $r = 1$: $\Delta$-systems [ES78, Kos00]
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Requirements over three strings

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  Tolhuizen’s solution concerning codes for a multiplying channel (2000).
Requirements over three strings

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- $r = 3$: Selective sets [EFF85]
Requirements over three strings

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  - Conflict resolution in multiaccess channel
Requirements over three strings

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  - Superimposed codes
Requirements over three strings

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  - Conflict resolution in multiaccess channel
  - Superimposed codes
  - Group Testing ...
Requirements over three strings

Problem

Determine the maximum cardinality of $C \subseteq [V]^n$ such that for any 3 of its elements there exists $r$ coordinates in which the respective columns of the strings are all different and of weight 1.

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  - Group Testing ...
Requirements over four strings

Problem

Find the maximum cardinality of $C \subseteq [V]^n$ such that for any 4 of its strings there are $r$ coordinates in which the respective columns of the strings are all different and of weight 1.
Requirements over four strings

**Problem**

Find the maximum cardinality of $C \subseteq [V]^n$ such that for any 4 of its strings there are $r$ coordinates in which the respective columns of the strings are all different and of weight 1.

- $r = 1$ : 4–locally thin sets [FKM01]
Requirements over four strings

**Problem**

Find the maximum cardinality of $C \subseteq [V]^n$ such that for any 4 of its strings there are $r$ coordinates in which the respective columns of the strings are all different and of weight 1.

- $r = 1$ : 4–locally thin sets [FKM01]
- $r = 2$ : 2–cancellative families [KS07]
Requirements over four strings

Problem

Find the maximum cardinality of \( C \subseteq [V]^n \) such that for any 4 of its strings there are \( r \) coordinates in which the respective columns of the strings are all different and of weight 1.

- \( r = 1 \) : 4–locally thin sets [FKM01]
- \( r = 2 \) : 2–cancellative families [KS07]
- \( r = 4 \) : Selective families
2–Cancellative Families

Theorem

\[ 0.11 \leq \limsup_{n \to +\infty} \frac{1}{n} \log M(n) \leq 0.42 \]
2–Cancellative Families

Theorem

\[
0.11 \leq \limsup_{n \to +\infty} \frac{1}{n} \log M(n) \leq 0.42
\]

Proof

Upper Bound: Use Tolhuizen’s result.
2–Cancellative Families

Theorem

\[
0.11 \leq \limsup_{n \to +\infty} \frac{1}{n} \log M(n) \leq 0.42
\]

Proof

Upper Bound: Use Tolhuizen’s result.

Lower Bound: Use random choice.
2–Cancellative Families

Theorem

\[
0.11 \leq \limsup_{n \to +\infty} \frac{1}{n} \log M(n) \leq 0.42
\]

Proof

**Upper Bound:** Use Tolhuizen’s result.

**Lower Bound:** Use random choice.
Conclusion

- $G$–differenti Permutation
- Forbiddance Problems
- Cancellativity
Conclusion

- $G$–differenti Permutation
- Forbiddance Problems
- Cancellativity

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Forbiddance Problems
2–Cancellative Families

Conclusion