

BRAIDING NON-ORIENTABLE SURFACES IN S^4

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Abstract

Closed braided surfaces in S^4 are the two-dimensional analogues of closed braids in S^3 . They are useful in studying smooth closed orientable surfaces in S^4 , since any such a surface is isotopic to a braided one. We show that the non-orientable version of this result does not hold, that is smooth closed non-orientable surfaces cannot be braided. In fact, any reasonable definition of non-orientable braided surfaces leads to very strong restrictions in terms of self-intersection and Euler characteristic.

Key words and phrases: non-orientable surfaces, braided surfaces in fibre bundles.

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Introduction

The concept of braided surface in $B^4 \cong B^2 \times B^2$ has been introduced since the early eighties by Rudolph (cf. [18], [19] and [20]) as a two dimensional analogue of the classical Artin's braids. Namely, he called braided a surface in $B^2 \times B^2$ which projects onto the first factor by a branched covering.

Successively, in the nineties, Viro and Kamada (cf. [8], [9] and [11]) considered closed braided surfaces in S^4 , that is surfaces contained in a normal neighborhood of $S^2 \subset S^4$, projecting onto S^2 by a branched covering. We can think of a closed braided surface as closure of a Rudolph's braided surface with trivial boundary, just in the same way we think of a closed braid in S^3 as closure of an Artin's braid.

By Kamada's results, closed braided surfaces can be used to study orientable smooth surfaces in S^4 . In fact, he provided two dimensional versions of the Alexander's and Markov's theorems on braids, by proving that any such a surface is isotopically equivalent to a closed braided surface and finding a set of moves relating isotopic braided surfaces.

In this paper we deal with the following question: can the above mentioned results be adapted in order to handle non-orientable surfaces in S^4 , replacing the standard 2-sphere as base model for braided surfaces with some standard non-orientable surface, such as the Veronese surface (see section 3)?

The question is relevant in relation to the representation of orientable closed smooth 4-manifolds as branched covers of S^4 , in which non-orientable surfaces play an essential role as branch sets in S^4 (see [6] and [17]).

Unfortunately, the answer is generally negative, in spite of some partial result obtained by Kamada (cf. [7]). In fact, in section 3 we show that there are very restrictive conditions for a non-orientable smooth surface in S^4 to be isotopic to a braided one. Nevertheless, we don't know whether any orientable smooth closed 4-manifold is a cover of S^4 branched over a (possibly non-orientable) braided surface.

In order to study non-orientable braided surfaces in S^4 , in section 2 we consider braided surfaces in R^2 -bundles over surfaces and prove a few of preliminary results about them, which are of some interest independently of the present application.

This paper is a revised version of part of the degree thesis [21] written by the second author under the supervision of the first author.

1. Preliminaries

To begin with, we reformulate in terms of coverings the classical Artin's notion of braid. By a *geometric braid* of degree d in R^3 we mean a 1-submanifold $b \subset [0, 1] \times R^2 \subset R^3$ such that the canonical projection $\pi : [0, 1] \times R^2 \rightarrow [0, 1]$ restricts to a covering $\pi|_b : b \rightarrow [0, 1]$ of degree d and moreover, putting $b_t = \{x \in R^2 \mid (t, x) \in b\}$, we have $b_0 = b_1 = *$ for a fixed $* = \{*_1, \dots, *_d\} \subset R^2$.

Considering braids of degree d up to fibre preserving (with respect to π) ambient isotopy of $[0, 1] \times R^2$, we can think of them as elements of the *braid group* $B_d = \pi_1(S_d R^2, *)$ of degree d , where $S_d R^2 \cong (\Pi_d R^2 - \Delta)/\Sigma_d$ denotes the space of all the subsets of R^2 consisting of d distinct points.

We recall the standard presentation of B_d (cf. [4]), with generators x_1, \dots, x_{d-1} , defined as shown in figure 1, and relations $x_i x_j = x_j x_i$ for any $i, j = 1, \dots, d-1$ such that $|i - j| > 1$ and $x_i x_{i+1} x_i = x_{i+1} x_i x_{i+1}$ for any $i = 1, \dots, d-2$.

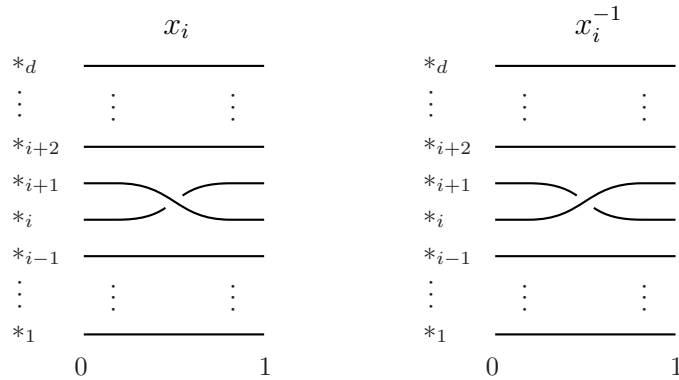


FIGURE 1.

Given a braid $b = x_{j_1}^{\varepsilon_1} \dots x_{j_k}^{\varepsilon_k} \in B_d$, we define the *index* of b to be the exponent sum $i(b) = \varepsilon_1 + \dots + \varepsilon_k$. Since all the relations above are balanced, it immediately follows that $i(b)$ is well-defined, that is it does not depend on the particular expression of b as a power product of standard generators.

We call a *closed braid* of degree d in R^3 any link $l \subset N(S^1) \subset R^3$, where $N(S^1)$ is a fixed open tubular neighborhood of S^1 in R^3 , such that the orthogonal projection $\pi : N(S^1) \cong S^1 \times R^2 \rightarrow S^1$ restricts to a covering $\pi|_l : l \rightarrow S^1$ of degree d . By Alexander's theorem, any link in R^3 is ambient isotopic to a closed braid.

The *closure* of a braid $b \in B_d$ is the closed braid $\widehat{b} = (\varphi \times \text{id}_{R^2})(b)$, where $\varphi : [0, 1] \rightarrow S^1$ is the usual parametrization given by $\varphi(t) = (\cos 2\pi t, \sin 2\pi t)$. Of course, \widehat{b} is defined only up to fibre preserving ambient isotopy of $N(S^1) \cong S^1 \times R^2$, being $b \in B_d$ defined only up to fiber preserving ambient isotopy of $[0, 1] \times R^2$. Viceversa, \widehat{b} uniquely determines b up to conjugation in B_d .

Then, it makes sense to define the index of a closed braid l in R^3 by putting $i(l) = i(b)$, where $b \in B_d$ is any braid such that $l = \widehat{b}$, being the index of braids obviously invariant under conjugation in B_d .

The index $i(l)$ of a closed braid l of degree d satisfies the following *Bennequin inequality* (cf. [3]), involving the Euler characteristic $\chi(S)$ of any surface $S \subset R^3$ such that $l = \text{Bd } S$ (that is a Seifert surface for l): $|i(l)| \leq d - \chi(S)$.

Finally, we recall the notion of branched covering between surfaces, which is needed in order to consider braided surfaces. A map $p : S \rightarrow X$ between compact surfaces is called a *branched covering* iff at any $s \in S$ it is locally equivalent to the complex map $z \mapsto z^{d(s)}$, where $d(s) \geq 1$ is the *local degree* of p at s . The *branch points* of p are the images of the singular ones, that is of the points $s \in S$ such that $d(s) > 1$. Moreover, p is called *simple* if $d(s) = 2$ for any singular point $s \in S$ and p is injective on the singular points.

2. Braided surfaces in fiber bundles

Let $f : N \rightarrow X$ be an R^2 -bundle over a compact connected surface X with (possibly empty) boundary. We call (*simple*) *braided surface* of degree d over X any locally flat compact surface $S \subset N$ such that the restriction $p = f|_S : S \rightarrow X$ is a (simple) branched covering of degree d . Moreover, we call *twist point* of S any singular point $t \in S$ of p and denote by $d(t) \geq 2$ the *local degree* of S at t , that is the local degree of p at t .

For any twist point $t \in S$, there exists a commutative diagram like the following, where: $C \subset N$ is a closed neighborhood of t , $D \subset X$ is a closed neighborhood of $p(t)$, h and k are homeomorphisms, $b_t \subset S^1 \times \text{Int } B^2$ is a closed braid of degree $d(t)$, $C(b_t) \subset B^2 \times B^2$ is the cone of b_t with vertex $(0, 0)$, π is the canonical projection on the first factor.

$$\begin{array}{ccccccc} t & \in & S \cap C & \subset & C & \xrightarrow{f|_C} & D \\ \downarrow & & \downarrow & & h \downarrow & & k \downarrow \\ (0, 0) & \in & C(b_t) & \subset & B^2 \times B^2 & \xrightarrow{\pi} & B^2 \end{array}$$

If N is oriented, we can assume that h is orientation preserving (with respect to the standard orientation of $B^2 \times B^2$). Moreover, fixed any local orientation of X at $p(t)$, we can also assume that k is orientation preserving (with respect to the standard orientation of B^2). With these two assumptions, b_t turns out to be uniquely determined up to braid isotopy, in such a way that we can define the *local index* $i(t)$ of S at t to be the integer number $i(b_t)$. In fact, it can be easily seen that $i(t)$ depends only on S and on the orientation of N , while it does not depend on the choice of the local orientation of X .

If t is a simple twist point of S , then, by local flatness, b_t coincides with the closure of one of the braids $x_1^{\pm 1} \in B_2$, so that $C(b_t)$ can be thought to have equation $w = z^2$ or $w = \bar{z}^2$ (depending on the sign of the exponent), with respect to the complex coordinates (w, z) of $B^2 \times B^2 \subset C^2$.

On the other hand, if t is a smooth twist point of S , then we get for $C(b_t)$ the equation $w = z^{d(t)}$ or $w = \bar{z}^{d(t)}$, while b_t turns out to be the closure of the braid $(x_1 \cdots x_{d(t)-1})^{\pm 1} \in B_{d(t)}$.

Hence, any simple twist point of S is smooth (up to fiber preserving ambient isotopy of N), and any smooth twist point t can be easily perturbed to get $d(t) - 1$ simple ones (up to ambient isotopy of N which does not preserve the fibers of f). In this case, we can assign to the twist point $t \in S$ a sign $s(t) = \pm 1$, depending only on the local shape of S and on the orientation of N , in such a way that $i(t) = s(t)(d(t) - 1)$.

For a non-smooth twist point t it may not exist any simple perturbation up to ambient isotopy, as it is shown in [10]. Nevertheless, we can modify S in a neighborhood of t in order to get a new braided surface S' , where the twist point t is replaced by a certain number of simple twist points t_1, \dots, t_k such that $i(t) = i(t_1) + \dots + i(t_k)$. Namely, if b_t is the closure of a braid $x_{j_1}^{\varepsilon_1} \cdots x_{j_k}^{\varepsilon_k} \in B_{d(t)}$, then we can replace $C(b_t)$ by a braided surface in $B^2 \times B^2$, having a positive (resp. negative) simple twist point t_l for each $\varepsilon_l = +1$ (resp. $\varepsilon_l = -1$), with $l = 1, \dots, k$ (cf. proposition 1.11 of [19]).

We remark that the braided surface we put in place of $C(b_t)$ is not necessarily a disk, however it is homologous to $C(b_t)$ mod the common boundary b_t . Then, S and S' may not be isotopic, but they share some homological properties, in particular they have the same self-intersection number as multi-valued sections of f . Moreover, by local flatness, the closed braid b_t represents the unknot, hence the Bennequin inequality implies that $|i(t)| \leq d(t) - 1$.

Still assuming N oriented, we define the *index* of S as the sum $i(S) = \sum i(t)$, where t runs over all the twist points of S . The following proposition gives the index of S in terms of its degree d and of the Euler number e of the bundle $f : N \rightarrow X$ over a closed surface X .

PROPOSITION 2.1. *If N is oriented and $S \subset N$ is any braided surface over a closed surface X as above, then $i(S) = ed(d - 1)$.*

Proof. By the definition of index and the previous observations, we can assume that S is simple, so that $i(S)$ equals the algebraic number of the twist points of S .

To begin with, we consider a handlebody decomposition of the base surface X , consisting of one 0-handle H^0 , 1-handles H_1^1, \dots, H_{2g+h}^1 attached to H^0 in the standard way depicted in figure 2, with $g, h \geq 0$ and H_j^1 orientable (resp. non-orientable) for $j = 1, \dots, 2g$ (resp. $j = 2g + 1, \dots, 2g + h$), and one 2-handle H^2 .

We can assume that all the branch points of p belong to $\text{Int } H^2$, in such a way that, putting $X_1 = H^0 \cup H_1^1 \cup \dots \cup H_{2g+h}^1$, $N_1 = f^{-1}(X_1)$ and $S_1 = S \cap N_1$, the restriction $p|_{S_1} : S_1 \rightarrow X_1$ is an ordinary covering.

By a suitable choice of the trivializations of f over the handles H^0 and H_j^1 , we can think of N_1 as the quotient space obtained by attaching $H_j^1 \times R^2$ to $H^0 \times R^2$ for all $j = 1, \dots, 2g + h$, by fiber preserving maps, whose restrictions to the fibers coincide with id_{R^2} or σ , where $\sigma : R^2 \rightarrow R^2$ is the symmetry with respect to the y -axis.

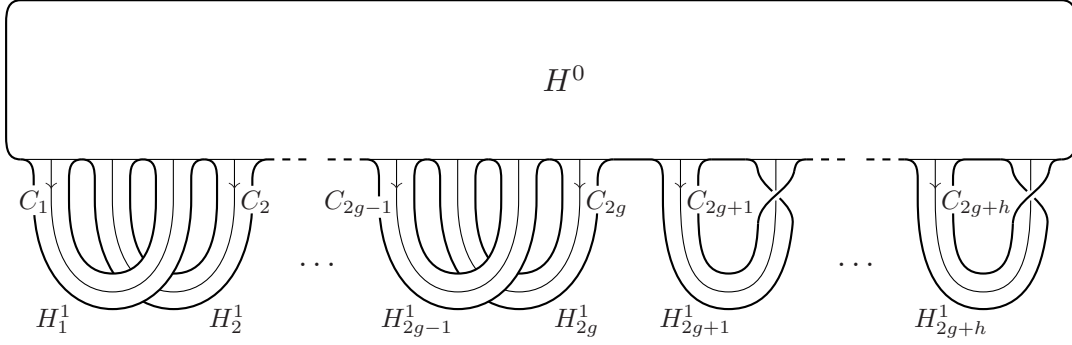


FIGURE 2.

Moreover, we can assume that the trivialization $f^{-1}(H^0) \cong H^0 \times R^2$, makes $S_0 = S \cap f^{-1}(H^0)$ into $H^0 \times \{*_1, \dots, *_d\} \subset H^0 \times R^2$, where $*_1, \dots, *_d$ belong to the x -axis and $\sigma(\{*_1, \dots, *_d\}) = \{*_1, \dots, *_d\}$. Then, for every $j = 1, \dots, 2g+h$, the trivialization $f^{-1}(H_j^1) \cong H_j^1 \times R^2$ makes $S \cap f^{-1}(C_j)$ into a braid $c_j \subset C_j \times R^2$, where C_j denote the core of the handle H_j , oriented as in figure 2.

We observe that S_1 is completely determined (up to fiber preserving isotopy) by the braids $c_1, \dots, c_{2g+h} \in B_d$ and $\text{Bd } S_1$ is a closed braid in $\text{Bd } N_1 \cong \text{Bd } X_1 \times R^2 \cong S^1 \times R^2$, which can be thought as the closure of the braid $c = c_1 c_2 c_1^{-1} c_2^{-1} \cdots c_{2g-1} c_{2g} c_{2g-1}^{-1} c_{2g}^{-1} c_{2g+1} c_{2g+1}^\sigma \cdots c_{2g+h} c_{2g+h}^\sigma \in B_d$, where c_j^σ denotes the image of c_j under the action of σ .

Putting $N_2 = f^{-1}(H^2)$ and $S_2 = S \cap N_2$, we have that $\text{Bd}(S_2) = \text{Bd}(S_1)$ is a closed braid in $\text{Bd } N_2 \cong \text{Bd } H^2 \times R^2 \cong S^1 \times R^2$, which can be thought as the closure of the braid $c' = ct^e$, where $t \in B_d$ denotes one positive full twist of d strings. On the other hand, denoting by $t_1, \dots, t_n \in S_2$ the (simple) twist points of S , it is straightforward (for example, see proposition 1.11 of [19]) to get $c' = y_1 x_{j_1}^{s(t_1)} y_1^{-1} \cdots y_n x_{j_n}^{s(t_n)} y_n^{-1}$, where each x_j is a standard generator of B_d and $s(t_j) = \pm 1$ as above.

At this point, we can finish the proof by observing that the computation of $i(c')$ based on the first expression of c' as a product of powers of generators gives us $ed(d-1)$, while the second one gives us $s(t_1) + \dots + s(t_k)$, that is $i(S)$. \square

As a consequence of proposition 2.1, we get numerical obstructions to the existence of braided surfaces, in terms of Euler characteristic and number of twist points.

PROPOSITION 2.2. *If N is oriented and $S \subset N$ is any braided surface over a closed surface X as above, then $\chi(S) \leq d(\chi(X) - |e|(d-1))$. Moreover, if S is simple, then the number of twist points is even and not less than $|e|d(d-1)$.*

Proof. By the Hurwitz formula we have $\chi(S) = d\chi(X) - \sum(d(t) - 1)$. Then, the first part of the proposition follows immediately by proposition 2.1 and by the inequalities $|i(S)| \leq \sum |i(t)| \leq \sum(d(t) - 1)$. For the second part, it is enough to observe that, if S is simple, then the number of twist points coincides with $\sum(d(t) - 1)$, which is congruent to $i(S) = \sum s(t)(d(t) - 1) \pmod{2}$. \square

Now, we want to show that the inequalities given by the proposition 2.2 are sharp. Given any R^2 -bundle $f : N \rightarrow X$ with oriented total space N and arbitrary

Euler number $e > 0$ (the case $e < 0$ can be covered by reversing the orientation of N), let $S_1, \dots, S_d \subset N$ be d smooth sections of f , transversally meeting each other in e points. Then, replacing each of the $ed(d-1)/2$ double points of $S_1 \cup \dots \cup S_d$ with one pair of positive simple twist points, as shown in figure 3, we get a simple braided surface S of degree d over X , with $ed(d-1)$ positive twist points and $\chi(S) = d(\chi(X) - e(d-1))$. On the other hand, we can easily add to S pairs of opposite simple twist points, as shown in figure 4, in order to arbitrarily increase the number of twist points of S and decrease the Euler characteristic $\chi(S)$.

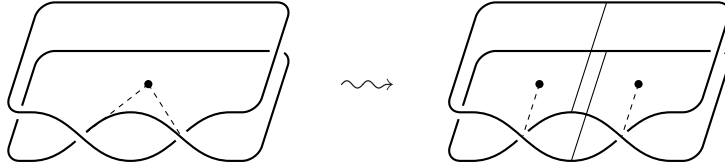


FIGURE 3.



FIGURE 4.

We conclude this section by computing the Euler number $e(S)$ of the braided surface S , that is the self-intersection number of S in the oriented 4-manifold N , which coincides with the self-intersection of S as a multi-valued section of f .

PROPOSITION 2.3. *If N is oriented and $S \subset N$ is any braided surface over a closed surface X as above, then $e(S) = i(S) + ed = ed^2$.*

Proof. Let $s : X \rightarrow N$ be a cross section of f transverse to the null section. We can assume that S is simple and that the zeroes of s are not branch points. By translating $s(x)$ at every point in $S \cap f^{-1}(x)$ for every $x \in X$ and taking normal component with respect to S , we get a normal vector field v along S with non-degenerate singularities.

A point $y \in S$ is a singular point for v iff $f(y)$ is a singular point for s or y is a twist point for S ; furthermore all the signs are coherent. Therefore we have $e(S) = i(S) + ed$. Then, the statement follows by proposition 2.1. \square

We notice that the results above can be easily generalized to the case of singular braided surfaces with transversal double points, by taking account of each double point as a pair of twist points. Namely, denoting by $n(S)$ the algebraic number of double points of S , we have $i(S) + 2n(S) = ed(d-1)$ and $e(S) = ed^2 - 2n(S)$.

3. Non-orientable braided surfaces in S^4

In this section we apply the results of the previous one, in order to show that the Viro-Kamada's representation theorem of orientable surfaces in S^4 as braided surfaces (cf. [8]) cannot be extended to include the non-orientable case.

In fact, by combining the results of the previous section with the Whitney's conjecture on non-orientable surfaces in S^4 , proved by Massey in [14], we get very restrictive conditions for such a surface to be isotopic to a braided one, with respect to any reasonable definition of non-orientable braided surface. We recall that the Whitney conjecture imposes the following constraints to the self-intersection number e of a non-orientable surface of Euler characteristic χ in S^4 : $e \equiv 2\chi \pmod{4}$ and $|e| \leq 4 - 2\chi$.

It is natural to call a *non-orientable braided surface* in S^4 any non-orientable surface $S \subset S^4$ which is contained as a braided surface over X in the normal fiber bundle $\nu : N \rightarrow X$ of some fixed standard smooth non-orientable surface $X \subset S^4$, where N is identified with an open tubular neighborhood of X in S^4 .

The most significant choice for X is the Veronese surface $V \subset S^4$ defined in the following way. First of all, we consider the space $\mathcal{M} \cong R^9$ of the 3×3 matrices over R with the inner product given by $\langle A, B \rangle = \text{tr}(AB^T)$ for all $A, B \in \mathcal{M}$, and the map $\varphi : S^2 \rightarrow \mathcal{M}$ defined by $\varphi(x) = x^T x$ for any $x \in S^2 \subset R^3$. Since $\varphi(y) = \varphi(x)$ iff $y = \pm x$, we get an induced embedding $\psi : P^2 \rightarrow \mathcal{M}$, where P^2 is thought as the quotient of S^2 by the action of the antipodal map $x \mapsto -x$. Then, we put $V = \psi(P^2)$ after having identified S^4 with the intersection of the unit sphere of \mathcal{M} with the affine subspace $\mathcal{L} = \{M \in \mathcal{M} \mid M = M^T \text{ and } \text{tr } M = 1\} \cong R^5$.

The remarkable property that characterizes V is the existence of a symmetric splitting $S^4 \cong \bar{N} \cup_f \bar{N}$, where \bar{N} is a closed tubular neighborhood of V in S^4 and f is an involution of $\text{Bd } \bar{N}$ onto itself (see [13] and [16]). Such splitting has several relevant geometric properties (cf. [1] and [16]), moreover, from a topological point of view, it is related to the identification of S^4 with the quotient of the complex projective plane under complex conjugation, being V the branch set of the canonical projection $CP^2 \rightarrow CP^2/\sim \cong S^4$ (cf. [12], [13] and [15]).

COROLLARY 3.1. *Any non-orientable braided surface $S \subset S^4$ of degree d over the Veronese surface $V \subset S^4$ satisfies the following conditions: $\chi(S) \leq d(3 - 2d)$, $i(S) = 2d(d - 1)$ and $e(S) = 2d^2$. As a consequence, the only surface S braided over V with $\chi(S) = 1$ is V itself (up to isotopy of braided surfaces) and there is no surface S braided over V with $\chi(S) = 0, -1, -3, -5, -7$.*

Proof. The first part of the corollary immediately follows from the results of the previous section, by taking into account that $e(V) = 2$. Now, the first inequality implies that: for $\chi(S) = 1$ we have $d = 1$, that is $S \cong V$; on the other hand, for $d \geq 2$ we have $\chi(S) \leq -2$; furthermore, we have $d \geq 3$, that gives us $\chi(S) \leq -9$, if we assume $\chi(S)$ odd and less than 1, since in this case also d is odd, because of the congruence $2d^2 \equiv 2\chi(S) \pmod{4}$. \square

We remark that, by the equation $e(S) = 2d^2$, any surface $S \subset S^4$ braided over V has positive self-intersection. In order to get negative (resp. vanishing) self-intersection numbers, one could consider surfaces braided over $V' = \alpha(V)$ (resp. $V \# V'$), where $\alpha : S^4 \rightarrow S^4$ is the antipodal map.

Moreover, it is worth observing that, in the non-vanishing cases, only few values of the self-intersection among the ones allowed by the Whitney conjecture are realized by surfaces braided over V or V' . In fact, the self-intersection number of such a surface, besides having the very special form $e(S) = \pm 2d^2$, is bounded by the inequality $|e(S)| \leq 9/4 - \chi(S) + 3/4 \sqrt{9 - 8\chi(S)}$, that can be derived from the inequality of corollary 3.1 by a straightforward computation.

However, the following problem remains still open: *is it possible to represent any orientable closed smooth 4-manifold as a cover of S^4 branched over a (possibly non-orientable) braided surface?*

References

- [1] K. Arslan and R. Ezentas, *Construction of focal Veronese surface in S^4* , Far East J. Math. Sci. **4** (1996), 133–136.
- [2] T.F. Banchoff and O. Johnson, *The normal Euler class and singularities of projections for polyhedral surfaces in 4-spaces*, Topology **37** (1998), 419–439.
- [3] D. Bennequin, *Entrelacements et équations de Pfaff*, Astérisque **107-108** (1983), 87–161.
- [4] J.S. Birman, *Braids, Link, and Mapping Class Groups*, Ann. of Math. Studies **82**, Princeton University Press 1974.
- [5] J.S. Carter and M. Saito, *Knot diagrams and braid theory in dimension 4*, in “Real and complex singularities”, Res. Notes Math. Ser. **333**, Pitman 1995, 112–147.
- [6] M. Iori and R. Piergallini, *Four-manifolds as covers of S^4 branched over smooth surfaces*, to appear.
- [7] S. Kamada, *Non-orientable surfaces in 4-space*, Osaka J. Math. **26** (1989), 367–385.
- [8] S. Kamada, *A characterization of groups of closed orientable surfaces in 4-space*, Topology **33** (1994), 113–122.
- [9] S. Kamada, *Alexander’s and Markov’s theorems in dimension four*, Bull. AMS **31** (1994), 64–67.
- [10] S. Kamada, *On braid monodromies of non-simple braided surfaces*, Math. Proc. Cambridge Phil. Soc. **120** (1996), 237–245.
- [11] S. Kamada, *Surfaces in 4-space: A view of normal forms and braidings*, in “Lectures at Knots ’96”, Knots & Everything 15, World Scientific 1997, 39–71.
- [12] N.H. Kuiper, *The quotient space of $CP(2)$ by complex conjugation is the 4-space*, Math. Ann. **208** (1974), 175–177.
- [13] T. Lawson, *Splitting S^4 on RP^2 via the branched cover of CP^2 over S^4* , Proc. Amer. Math. Soc. **86** (1982), 328–330.
- [14] W.S. Massey, *Proof of a conjecture of Whitney*, Pacific J. Math. **31** (1969), 143–156.

- [15] W.S. Massey, *The quotient space of the complex projective plane under conjugation is a 4-sphere*, *Geom. Dedicata* **2** (1973), 371–374.
- [16] W.S. Massey, *Imbeddings of projective planes and related manifolds in spheres*, *Indiana Univ. Math. J.* **23** (1973/74), 791–812.
- [17] R. Piergallini, *Four-manifolds as 4-fold branched covers of S^4* , *Topology* **34** (1995), 497–508.
- [18] L. Rudolph, *Algebraic function and closed braids*, *Topology* **22** (1983), 191–202.
- [19] L. Rudolph, *Braided surfaces and Seifert ribbons for closed braids*, *Comment. Math. Helvetici* **58** (1983), 1–37.
- [20] L. Rudolph, *Special position for surfaces bounded by closed braids*, *Rev. Mat. Iberoamer.* **1** (1985), rev. marzo 2000.
- [21] D. Zuddas, *Trecce bidimensionali*, Grad. thesis, University of Camerino–Italy 2000.