

# AUTOMORPHISMS OF TRIVALENT GRAPHS

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## Abstract

Let  $\mathcal{G}_{g,b}$  be the set of all uni/trivalent graphs representing the combinatorial structures of pant decompositions of the oriented surface  $\Sigma_{g,b}$  of genus  $g$  with  $b$  boundary components. We describe the set  $\mathcal{A}_{g,b}$  of all automorphisms of graphs in  $\mathcal{G}_{g,b}$  showing that, up to suitable moves changing the graph within  $\mathcal{G}_{g,b}$ , any such automorphism can be reduced to elementary switches of adjacent edges.

*Keywords:* uni/trivalent graph,  $F$ -move, pant decomposition complex.

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## INTRODUCTION

Let  $\mathcal{G}_{g,b}$  be the set of all connected non-oriented uni/trivalent graphs  $\Gamma$ , with first Betti number  $\beta_1(\Gamma) = g \geq 0$ , at least one trivalent vertex and  $b \geq 0$  univalent vertices. It is immediate to see that such a graph  $\Gamma$  exists if and only if  $(g, b) \neq (0, 0), (0, 1), (0, 2), (1, 0)$ , and in this case  $\Gamma$  has  $2g - 2 + b$  trivalent vertices and  $3g - 3 + 2b$  edges. In the following, the pair  $(g, b)$  will be always assumed to satisfy that restriction.

Given any  $\Gamma \in \mathcal{G}_{g,b}$ , we denote by  $\mathcal{A}(\Gamma)$  the group of all the automorphisms of  $\Gamma$  as a non-oriented graph. In this paper we are interested in the structure of the set  $\mathcal{A}_{g,b} = \cup_{\Gamma \in \mathcal{G}_{g,b}} \mathcal{A}(\Gamma)$  of all the automorphisms of graphs in  $\mathcal{G}_{g,b}$ . More precisely, we will prove that, up to certain  $F$ -moves changing the graph within  $\mathcal{G}_{g,b}$ , any such automorphism can be reduced to elementary switches of adjacent edges.

The drive for this paper comes from a problem we met while working on pant decompositions of surfaces in [2]. Namely, in that paper we build up an infinite simply connected 2-dimensional complex  $\mathcal{R}_{g,b}$ , codifying all the pant decompositions on the connected compact oriented surface  $\Sigma_{g,b}$  of genus  $g$  with  $b$  boundary components, as well as all the moves relating them and all the relations between those moves. The construction is subdivided into two independent steps.

First a finite simply connected 2-dimensional complex  $\mathcal{S}_{g,b}$  is described, which codifies the *combinatorial structures* of pant decompositions of  $\Sigma_{g,b}$ . The combinatorial structure of a pant decomposition  $D$  consists of the information concerning

only the incidence relations between pants, and it is encoded by the dual graph  $\Gamma_D \in \mathcal{G}_{g,b}$ , with the trivalent vertices corresponding to the pants, the univalent vertices corresponding to the boundary components, and the edges corresponding to the cutting curves. Afterwards, by using a presentation of the mapping class group  $\mathcal{M}_{g,b}$  of  $\Sigma_{g,b}$ , we get the desired complex  $\mathcal{R}_{g,b}$ .

Due to technical issues, the switch from  $\mathcal{S}_{g,b}$  to  $\mathcal{R}_{g,b}$  is not a “direct” one, but it requires instead a cumbersome intermediate step. This involves in the construction of two additional complexes  $\widetilde{\mathcal{S}}_{g,b}$  and  $\widetilde{\mathcal{R}}_{g,b}$ , codifying respectively the *decorated combinatorial structures* and the *decorated pant decompositions*. Here, a decoration of a pant decomposition  $D$  is a numbering of its cutting curves, and a decoration of the corresponding combinatorial structure is a numbering of the edges of  $\Gamma_D$ .

This method allows us to sidestep a crucial problem in the direct transition from  $\mathcal{S}_{g,b}$  to  $\mathcal{R}_{g,b}$ , that is the study of the stabilization subgroup  $\text{Stab}(D) \in \mathcal{M}_{g,b}$  of any given pant decomposition  $D$ . In fact, the proof of the simple connectedness of  $\mathcal{R}_{g,b}$  given in [2] would become less elaborate based on the fact that the fiber  $p^{-1}(\Gamma_D)$  of the natural projection  $p : \mathcal{R}_{g,b} \rightarrow \mathcal{S}_{g,b}$  is trivial at  $\mathcal{R}_{g,b}$  on the level of the fundamental group. In other words, we need to show that any loop based at  $D$  and contained in  $p^{-1}(\Gamma_D)$  is contractible in  $\mathcal{R}_{g,b}$ . Now, such a loop corresponds to an element of  $\text{Stab}(D)$ , that is a symmetry of  $D$ , which induces a (possibly trivial) combinatorial symmetry of  $\Gamma_D$ . Actually, once the combinatorial symmetries of  $\Gamma_D$  are known, one can reconstruct those of  $D$  in a straightforward way, by adding some easy topological information.

At this point, it should be clear the motivation for the study carried out in the present paper of the structure of  $\mathcal{A}_{g,b}$ , and in particular of the inclusion  $\mathcal{A}(\Gamma) \subset \mathcal{A}_{g,b}$  of the group of combinatorial symmetries  $\mathcal{A}(\Gamma)$  of a given  $\Gamma \in \mathcal{G}_{g,b}$ . In fact, the result we obtain here will enable us to give a much simpler construction of  $\mathcal{R}_{g,b}$  than in [2], and a more direct proof of its simple connectedness starting from  $\mathcal{S}_{g,b}$  without involving decorations.

However, the study of the automorphisms of uni/trivalent graphs has its own interest independently from that specific application (see the classical papers [11, 12] and [9], and some more recent ones, as for instance [5, 4, 7, 8]). Hence, in this paper we focus on the structure of  $\mathcal{A}_{g,b}$ , while the new construction of  $\mathcal{R}_{g,b}$  mentioned above will be described in details in a forthcoming paper [3].

Moreover, we emphasize that our result may be easily restated in term of rotors, i.e. subgraph of  $G$  having rotational symmetry, and whose vertices of attachment are equivalent with respect to this symmetry. Rotors were introduced in graph theory by Tutte in [13], and the concept was adapted to knot theory as a generalization of Conway mutation (see for instance [1] and [6]). A careful analysis of those applications, in the light of our results, will be the object of further investigations.

The paper is structured as follows. In Section 1 we give the basic definitions, the statement of the main theorem, and the reduction of its proof to special cases. Then, after having established some preliminary results in Section 2, we devote the subsequent Sections 3, 4, 5 to those special cases.

## 1. DEFINITIONS AND MAIN THEOREM

For a graph  $\Gamma \in \mathcal{G}_{g,b}$ , we call a *free end* of  $\Gamma$  any univalent vertex, a *terminal edge* of  $\Gamma$  any of the  $b$  edges connecting a free end to a trivalent vertex, and an *internal edge* of  $\Gamma$  any of the  $3g - 3 + b$  edges connecting two (possibly coinciding) trivalent vertices.

First of all, we introduce  $F$ -moves on uni/trivalent graphs. These are well-known moves that change the graph structure by acting on the internal edges. It is a folklore result that  $F$ -moves, even in the most restrictive form given in Definition 1.1, suffice to relate any two graphs in  $\mathcal{G}_{g,b}$  up to graph isomorphisms (cf. [10]). However, we need to consider an invariant version of  $F$ -moves, in order to relate graph automorphisms. This makes things more involved.

**Definition 1.1.** Let  $\Gamma \in \mathcal{G}_{g,b}$  and  $e \in \Gamma$  be an internal edge with distinct ends. Then, we call *elementary (edge)  $F$ -move*, the modification  $F_{e,e'} : \Gamma \rightsquigarrow \Gamma'$  that makes  $\Gamma$  into  $\Gamma' \in \mathcal{G}_{g,b}$ , by replacing  $e$  with a new internal edge  $e'$  as in Figure 1, while leaving the rest of the graph unchanged. Clearly, the inverse modification  $F_{e',e} : \Gamma' \rightsquigarrow \Gamma$  is an elementary (edge)  $F$ -move as well.

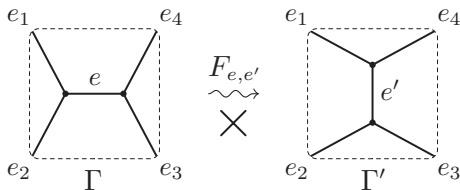


FIGURE 1. An edge  $F$ -move  $F_{e,e'}$ .

According to the literature, here the letter  $F$  stands for “fusion” and refers to the fact that the  $F$ -move  $F_{e,e'}$  can be thought as the contraction of the edge  $e$  in  $\Gamma$ , and the consequent fusion of its ends, followed by the inverse of a similar contraction of the edge  $e'$  in  $\Gamma'$ . The intermediate graph has a quadrivalent vertex in place of the two trivalent ends of the contracted edges, as indicated under the arrow.

We warn the reader that, since the graphs in  $\mathcal{G}_{g,b}$  are abstract ones, properties and constraints depending on the planar representation (of portions) of them in the figures do not have any significance.

In particular, the coupling  $(e_1e_2)(e_3e_4)$  of the (possibly not all distinct) edges  $e_1, \dots, e_4$  determined by the two ends of  $e$  in  $\Gamma$ , which in Figure 1 is replaced by the coupling  $(e_1e_4)(e_2e_3)$  determined by the two ends of  $e'$  in  $\Gamma'$ , could be replaced by the coupling  $(e_1e_3)(e_2e_4)$  as well. In other words, there are exactly two possible ways to perform the  $F$ -move  $F_{e,e'}$  at the given edge  $e$  of  $\Gamma$ , corresponding to the two possible changes of coupling, depending on the structure of  $\Gamma'$  at the edge  $e'$ :

$$(e_1e_2)(e_3e_4) \overset{F_{e,e'}}{\rightsquigarrow} (e_1e_4)(e_2e_3) \quad \text{and} \quad (e_1e_2)(e_3e_4) \overset{F_{e,e'}}{\rightsquigarrow} (e_1e_3)(e_2e_4).$$

From a different perspective, the elementary  $F$ -move described in Figure 1, can be interpreted as the replacement of a uni/trivalent subtree  $T \subset \Gamma$  with a different uni/trivalent subtree  $T' \subset \Gamma'$  having the same univalent vertices. Namely,  $T$  consists

of the edges  $e_1, \dots, e_4$  and  $e$ , while  $T'$  consists of the edges  $e_1, \dots, e_4$  and  $e'$ . This suggests the following generalization of the notion of elementary  $F$ -move.

**Definition 1.2.** Let  $\Gamma \in \mathcal{G}_{g,b}$  and  $T \subset \Gamma$  be a uni/trivalent subtree with  $m \geq 4$  free ends. Then, we call *elementary (tree)  $F$ -move*, the modification  $F_{T,T'} : \Gamma \rightsquigarrow \Gamma'$  that makes  $\Gamma$  into  $\Gamma' \in \mathcal{G}_{g,b}$ , by replacing  $T$  with a different uni/trivalent tree  $T' \subset \Gamma'$  having the same free ends of  $T$ , while leaving the rest of the graph unchanged (see Figure 2 for an example with  $m = 5$ ). Also in this case, the inverse modification  $F_{T',T} : \Gamma' \rightsquigarrow \Gamma$  is an elementary (tree)  $F$ -move as well.

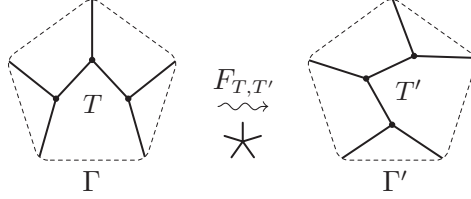


FIGURE 2. A tree  $F$ -move  $F_{T,T'}$ .

We note that the  $F$ -move  $F_{T,T'}$  is again a kind of fusion move. In fact, it can be thought as the contraction of the subtree consisting of all the internal edges of  $T \subset \Gamma$  to a single  $m$ -valent vertex, and the consequent fusion of all the trivalent vertices of  $T$ , followed by the inverse of a similar contraction for  $T' \subset \Gamma'$ .

If the subtree  $T \subset \Gamma$  has the vertices  $v_1, \dots, v_m$  as its free ends, then all the possible ways to perform the  $F$ -moves  $F_{T,T'}$  on  $\Gamma$  corresponds to all the complete iterated couplings of the (unordered) set  $\{v_1, \dots, v_m\}$  induced by  $T'$ , except the one induced by  $T$ . In particular, according to the above discussion, we have 2 ways for  $m = 4$ .

Actually, any tree  $F$ -move  $F_{T,T'}$  can be realized by a suitable sequence of edge  $F$ -moves performed on the edges of  $T$  (and of the corresponding subtrees in the resulting graphs). Yet, it makes sense to consider  $F_{T,T'}$  as a unique move, and we will see shortly why (cf. discussion about the  $F$ -move in Figure 3 below).

**Definition 1.3.** Let  $\Gamma \in \mathcal{G}_{g,b}$  and  $\mathcal{T} = \{T_1, \dots, T_k\}$  be a family of uni/trivalent subtrees of  $\Gamma$  with pairwise disjoint interiors (that is, two of them possibly share only some common free ends). Assuming that each  $T_i$  has at least 4 free ends, we call  *$F$ -move* any modification  $F_{\mathcal{T},\mathcal{T}'} : \Gamma \rightsquigarrow \Gamma'$  given by the simultaneous application of certain elementary moves  $F_{T_1,T'_1}, \dots, F_{T_k,T'_k}$  to  $\Gamma$ . Of course, for  $k = 1$  we have an elementary  $F$ -move.

For any graph  $\Gamma \in \mathcal{G}_{g,b}$  we denote by  $\mathcal{A}(\Gamma)$  the group of all the automorphisms of  $\Gamma$  as a non-oriented graph, including those that permute the free ends. Moreover, we denote by  $\mathcal{A}_{g,b} = \cup_{\Gamma \in \mathcal{G}_{g,b}} \mathcal{A}(\Gamma)$  the set of all the automorphisms of graphs in  $\mathcal{G}_{g,b}$ .

Now, assume we are given an automorphism  $\varphi \in \mathcal{A}(\Gamma)$ . If  $F_{\mathcal{T},\mathcal{T}'} : \Gamma \rightsquigarrow \Gamma'$  is an  $F$ -move with  $\mathcal{T} = \{T_1, \dots, T_k\}$  a  $\varphi$ -invariant family of subtrees of  $\Gamma$ , then the restriction of  $\varphi$  to  $\Gamma - \cup_i \text{Int } T_i$  induces a graph automorphism  $\psi$  of  $\Gamma' - \cup_i \text{Int } T'_i$ , under the canonical isomorphism  $\Gamma - \cup_i \text{Int } T_i \cong \Gamma' - \cup_i \text{Int } T'_i$  of graphs given by  $F_{\mathcal{T},\mathcal{T}'}$ . In general, such  $\psi$  does not extend to an automorphism  $\varphi' \in \mathcal{A}(\Gamma')$ , but if it does then the extension  $\varphi' : \Gamma' \rightarrow \Gamma'$  can be easily seen to be unique, by taking into account

that  $\Gamma - \cup_i \text{Int } T_i$  contains all the free ends of all the subtrees  $T_i$ . When  $\varphi'$  exists, we say that it is induced by  $\varphi$  through  $F_{\mathcal{T}, \mathcal{T}'}$ .

**Definition 1.4.** Given an automorphism  $\varphi : \Gamma \rightarrow \Gamma$ , we say that an  $F$ -move  $F_{\mathcal{T}, \mathcal{T}'} : \Gamma \rightsquigarrow \Gamma'$  is  $\varphi$ -invariant if  $\mathcal{T}$  is a  $\varphi$ -invariant family of subtrees and  $\varphi$  induces an automorphism  $\varphi' : \Gamma' \rightarrow \Gamma'$  through  $F_{\mathcal{T}, \mathcal{T}'}$  as discussed above. In this case, we write  $F_{\mathcal{T}, \mathcal{T}'} : \varphi \rightsquigarrow \varphi'$ . Such relation between automorphisms is symmetric, in the sense that  $F_{\mathcal{T}', \mathcal{T}} : \varphi' \rightsquigarrow \varphi$  holds as well, being  $F_{\mathcal{T}', \mathcal{T}} : \Gamma' \rightsquigarrow \Gamma$  a  $\varphi'$ -invariant  $F$ -move. But it is neither reflexive nor transitive. Then, we call  $F$ -equivalence the generated equivalence relation on the set  $\mathcal{A}_{g,b}$ .

We observe that the elementary  $F$ -move in Figure 1 is  $\varphi$ -invariant with respect to any automorphism  $\varphi \in \mathcal{A}(\Gamma)$  that acts on the depicted portion of  $\Gamma$  as any planar symmetry (horizontal, vertical or central). On the contrary, it is not  $\varphi$ -invariant for a  $\varphi$  that switches  $e_1$  and  $e_2$  while leaving  $e_3$  and  $e_4$  fixed.

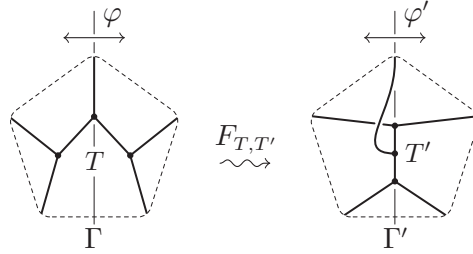


FIGURE 3. A  $\varphi$ -invariant  $F$ -move.

Figure 3 shows a  $\varphi$ -invariant elementary  $F$ -move  $F_{\mathcal{T}, \mathcal{T}'}$ , where  $\varphi \in \mathcal{A}(\Gamma)$  is any automorphism that acts on  $T$  as the symmetry with respect to the vertical axis. As we said above,  $F_{\mathcal{T}, \mathcal{T}'}$  could be realized by a sequence of elementary edge  $F$ -moves, but this is not true if we insist that each single edge  $F$ -move is  $\varphi$ -invariant. Finally, notice that instead the  $F$ -move described in Figure 2 is not  $\varphi$ -invariant (for the same automorphism  $\varphi$ ).

**Remark 1.5.** In the following, we will essentially use only  $\varphi$ -invariant  $F$ -moves  $F_{\mathcal{T}, \mathcal{T}'}$  consisting of simultaneous elementary edge  $F$ -moves  $F_{e, e'}$  or tree  $F$ -moves  $F_{\mathcal{T}, \mathcal{T}'}$  like the one in Figure 3. We emphasize once again that the  $\varphi$ -invariance of  $F$ -moves  $F_{\mathcal{T}, \mathcal{T}'}$  does not imply the  $\varphi$ -invariance of each single elementary  $F$ -move, being these possibly permuted by  $\varphi$ . In many cases, it will be convenient to think of those elementary moves as the edges slidings shown in Figure 4. Here, the edges  $f, f_1$  and  $f_2$  are the original ones of  $\Gamma$ , whose ends in the figure are slid to get their new positions in  $\Gamma'$  as indicated by the arrows, while the rest of the graph is fixed.

At this point, in order to state our main theorem, we are left to introduce the elementary automorphisms, which will generate all the automorphisms in  $\mathcal{A}_{g,b}$  up to  $F$ -equivalence.

**Definition 1.6.** By an *elementary automorphism* of a graph  $\Gamma \in \mathcal{G}_{g,b}$  we mean any automorphism  $S_{e_1, e_2} : \Gamma \rightarrow \Gamma$  interchanging two adjacent edges  $e_1$  and  $e_2$  of  $\Gamma$ , while fixing all the rest of the graph. We call such an automorphism  $S_{e_1, e_2} \in \mathcal{A}(\Gamma)$  a *terminal switch* or an *internal switch*, depending on the fact that  $e_1$  and  $e_2$  are terminal or internal edges (cf. Figure 5).

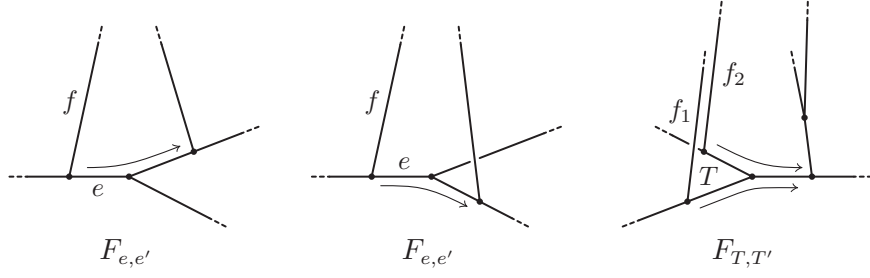


FIGURE 4.  $F$ -moves as edge slidings.

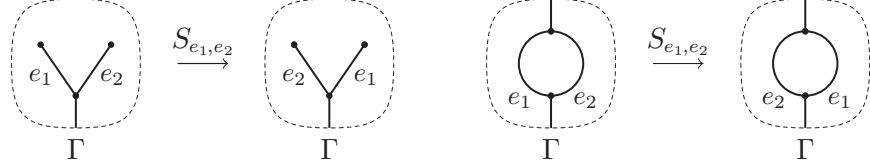


FIGURE 5. A terminal switch (left side) and an internal switch (right side).

**Remark 1.7.** An automorphism in  $\Gamma \in \mathcal{G}_{g,b}$  is uniquely determined up to internal switches by its action on the vertices of  $\Gamma$ . Moreover, it is uniquely determined up to (terminal and internal) switches by its action on the trivalent vertices of  $\Gamma$ .

To simplify our claims, we provide the next definition.

**Definition 1.8.** For any  $g$  and  $b$ , we denote by  $\mathcal{E}_{g,b} \subset \mathcal{A}_{g,b}$  the smallest subset that contains all the elementary automorphisms (internal and terminal switches) in  $\mathcal{A}_{g,b}$  and is closed with respect to composition and  $F$ -equivalence of automorphisms.

Then, our main result can be stated as follows.

**Theorem 1.9.** For any  $g$  and  $b$ , we have  $\mathcal{E}_{g,b} = \mathcal{A}_{n,b}$ .

*Proof.* We have to show that any element  $\varphi \in \mathcal{A}_{g,b}$ , that is any automorphism  $\varphi : \Gamma \rightarrow \Gamma$  of a graph  $\Gamma \in \mathcal{G}_{g,b}$ , actually belongs to  $\varphi \in \mathcal{E}_{g,b}$ . By primary decomposition of the cyclic subgroup  $\langle \varphi \rangle \subset \mathcal{A}(\Gamma)$ , we can reduce ourselves to the special case when the order  $\text{ord}(\varphi)$ , that is the cardinality of  $\langle \varphi \rangle$ , is a prime power. Then, the subcases when  $\text{ord}(\varphi) = p^m$  with  $p$  prime  $> 3$  or  $\text{ord}(\varphi) = 3^m$  are reduced to the case  $\text{ord}(\varphi) = 2$  in Sections 3 and 4 respectively. Finally, the case of  $\text{ord}(\varphi) = 2^m$  is proved in Section 5.  $\square$

## 2. SOME PRELIMINARY RESULTS

Let  $\varphi : \Gamma \rightarrow \Gamma$  be an automorphism of a uni/trivalent graph  $\Gamma$ . Two elements (vertices or edges) or subgraphs  $x$  and  $y$  of  $\Gamma$  are said to be  $\varphi$ -equivalent if  $y = \varphi^i(x)$  for some  $i \geq 0$ . In this case we write  $x \cong_\varphi y$ .

We denote by  $\text{ord}(\varphi)$  the order of  $\varphi$  in  $\mathcal{A}(\Gamma)$  (as an automorphism of a non-oriented graph), that is the cardinality of  $\langle \varphi \rangle \subset \mathcal{A}(\Gamma)$ . For a vertex  $v$  of  $\Gamma$ , by the *order of  $v$  with respect to  $\varphi$* , we mean as usual the cardinality  $\text{ord}_\varphi(v)$  of its  $\varphi$ -orbit  $\text{Orb}_\varphi(v) = \{\varphi^k(v) \mid 0 \leq k < \text{ord}(\varphi)\}$ . On the contrary, for an edge  $e$  of  $\Gamma$  with (possibly coinciding) ends  $v$  and  $w$ , we call *order of  $e$  with respect to  $\varphi$*  the number  $\text{ord}_\varphi(e) = \text{LCM}(\text{ord}_\varphi(v), \text{ord}_\varphi(w))$ , which is equal to either the cardinality

of the  $\varphi$ -orbit  $\text{Orb}_\varphi(e) = \{\varphi^k(e) \mid 0 \leq k < \text{ord}(\varphi)\}$  or its double (if  $v \neq w$  and  $(v, w) \cong_\varphi (w, v)$  as ordered pairs). This amounts to consider  $e$  as an oriented edge, except the case when it is a loop.

**Lemma 2.1.** *If  $\varphi$  and  $\varphi'$  are  $F$ -equivalent in  $\mathcal{A}_{g,b}$ , then  $\text{ord}(\varphi) = \text{ord}(\varphi')$ .*

*Proof.* It suffices to consider the case when  $\varphi$  and  $\varphi'$  are related by a single  $F$ -move  $F_{\mathcal{T}, \mathcal{T}'} : \varphi \rightsquigarrow \varphi'$ , with  $\mathcal{T} = \{T_1, \dots, T_k\}$  and  $\mathcal{T}' = \{T'_1, \dots, T'_k\}$ . Then,  $\text{ord}(\varphi)$  coincides with the order of its restriction to  $\Gamma - \cup_i \text{Int } T_i$ , by the uniqueness of extension to the subtrees  $T_i$ . Similarly,  $\text{ord}(\varphi')$  coincides with the order of its restriction to  $\Gamma' - \cup_i \text{Int } T'_i$ . Since those restrictions coincide under the canonical isomorphism  $\Gamma - \cup_i \text{Int } T_i \cong \Gamma' - \cup_i \text{Int } T'_i$ , we have  $\text{ord}(\varphi) = \text{ord}(\varphi')$ .  $\square$

**Lemma 2.2.** *Let  $\varphi : \Gamma \rightarrow \Gamma$  be an automorphism of a (possibly disconnected) uni/trivalent graph  $\Gamma$ . If  $v$  is a vertex of  $\Gamma$  and  $e$  is an edge of  $\Gamma$  having  $v$  as an end, then we have one of the following:*

- 1)  $\text{ord}_\varphi(e) = \text{ord}_\varphi(v)$ ;
- 2)  $\text{ord}_\varphi(e) = 2 \text{ord}_\varphi(v)$ , which happens if and only if  $v$  is a trivalent vertex, with three distinct edges  $e, e'$  and  $e''$  exiting from it, such that  $e' = \varphi^{\text{ord}_\varphi(v)}(e)$  and  $\text{ord}_\varphi(e'') = \text{ord}_\varphi(v)$ ;
- 3)  $\text{ord}_\varphi(e) = 3 \text{ord}_\varphi(v)$ , which happens if and only if  $v$  is a trivalent vertex, with three distinct edges  $e, e'$  and  $e''$  exiting from it, such that  $e' = \varphi^{\text{ord}_\varphi(v)}(e)$  and  $e'' = \varphi^{2 \text{ord}_\varphi(v)}(e)$ .

Moreover, if  $\Gamma$  is connected then the set of all orders of edges is  $\{m, 2m, \dots, 2^k m\}$  for some  $m \geq 1$  and  $k \geq 0$  such that  $\text{ord}(\varphi) = 2^k m$ , while the set of all orders of vertices is  $\{m, 2m, \dots, 2^h m, 2^{j_1} m/3, \dots, 2^{j_\ell} m/3\}$  with  $k-1 \leq h \leq k$ ,  $0 \leq l \leq k$ ,  $0 \leq j_1 < \dots < j_\ell \leq k$ , and  $m$  multiple of 3 if  $\ell > 0$ .

*Proof.* By definition of order for edges, we immediately have that  $\text{ord}_\varphi(e)$  is a multiple of  $\text{ord}_\varphi(v)$ . Moreover, since  $\varphi^{i \text{ord}_\varphi(v)}(e)$  is an edge exiting from  $\varphi^{i \text{ord}_\varphi(v)}(v) = v$  for any  $i \geq 1$ , the only possible cases are  $\text{ord}_\varphi(e) = \text{ord}_\varphi(v), 2 \text{ord}_\varphi(v), 3 \text{ord}_\varphi(v)$  and these are verified according to the conditions given in the statement. As a consequence, if the orders of two adjacent edges are different, then one of them is the double of the other, being those edges like  $e$  and  $e''$  in point 2. For  $\Gamma$  connected, this implies that the set of orders of the edges is  $\{m, 2m, \dots, 2^k m\}$  for some  $m \geq 1$  and  $k \geq 0$ . In order to get the set of orders of vertices, it suffices to observe that if  $e$  and  $e'$  are two edges exiting from  $v$ , such that  $\text{ord}_\varphi(e') = 2 \text{ord}_\varphi(e)$ , then  $\text{ord}_\varphi(v) = \text{ord}_\varphi(e)$ , which gives the orders  $m, 2m, \dots, 2^h m$  with  $k-1 \leq h \leq k$ . On the other hand, tripodes of edges whose order has the form  $2^j m/3$  may appear, like in point 3, for any  $j = 0, \dots, k$ .  $\square$

By a *path*  $\alpha$  between the (possibly coinciding) vertices  $v$  and  $w$  in a graph  $\Gamma$  we mean a (possibly non-simple) chain of edges having  $v$  and  $w$  as its ends. We call the number  $\text{len}(\alpha)$  of (non-necessarily distinct) edges in the chain the *length* of  $\alpha$ . Moreover, we denote by  $\bar{\alpha}$  the reversed path.

The next four lemmas concern minimal paths between vertices in a given  $\varphi$ -orbit.

**Lemma 2.3.** *Let  $\varphi : \Gamma \rightarrow \Gamma$  be an automorphism of a (possibly disconnected) uni/trivalent graph  $\Gamma$ . Given any (simple) path  $\alpha \subset \Gamma$  of minimal length among all*

paths joining any two different vertices of a given  $\varphi$ -orbit, let  $v$  and  $v'$  the ends of  $\alpha$  in that  $\varphi$ -orbit. If  $i \neq j$  and the four points  $\varphi^i(v)$ ,  $\varphi^i(v')$ ,  $\varphi^j(v)$  and  $\varphi^j(v')$  are all distinct, then the paths  $\varphi^i(\alpha)$  and  $\varphi^j(\alpha)$  are disjoint.

*Proof.* Let  $\alpha$  be a path as in the statement. By contradiction, assume  $\varphi^i(\alpha)$  and  $\varphi^j(\alpha)$  are not disjoint for some  $i \neq j$  satisfying the required condition. Then, they share at least a common internal edge  $e$ , and we have one of the two situations depicted in Figure 6, depending on the fact that  $e$  is traversed in the same direction or not when going from  $\varphi^i(v)$  to  $\varphi^i(v')$  along  $\varphi^i(\alpha)$  and from  $\varphi^j(v)$  to  $\varphi^j(v')$  along  $\varphi^j(\alpha)$ . Notice that the paths  $\beta_{i,1}$  and  $\beta_{i,2}$  forming  $\varphi^i(\alpha) - \text{Int}(e)$  are not necessarily disjoint from the paths  $\beta_{j,1}$  and  $\beta_{j,2}$  forming  $\varphi^j(\alpha) - \text{Int}(e)$ . In any case, we find a path shorter than  $\alpha$  between different vertices in the given  $\varphi$ -orbit. Namely, such path is either  $\beta_{i,1}\bar{\beta}_{j,1}$  or  $\beta_{i,2}\bar{\beta}_{j,2}$  in the left side case, while it is either  $\beta_{i,1}\bar{\beta}_{j,2}$  or  $\beta_{i,2}\bar{\beta}_{j,1}$  in the right side case.  $\square$

**Lemma 2.4.** *Let  $\varphi : \Gamma \rightarrow \Gamma$ ,  $\alpha \subset \Gamma$ ,  $v$  and  $v'$ , be as in Lemma 2.3. If  $i \neq j$  with  $\varphi^j(v) = \varphi^i(v')$  and  $\varphi^j(v') \neq \varphi^i(v)$ , then the union  $\varphi^i(\alpha) \cup \varphi^j(\alpha)$  has the structure shown in Figure 7, where: 1)  $k \geq 1$ ; 2) the paths  $\delta_{i,1}, \dots, \delta_{i,k} \subset \varphi^i(\alpha)$  are disjoint from the paths  $\delta_{j,1}, \dots, \delta_{j,k} \subset \varphi^j(\alpha)$ ; 3)  $\text{len}(\delta_{i,h}) = \text{len}(\delta_{j,k-h+1}) \geq 1$  for every  $h = 1, \dots, k$ ; 4)  $\text{len}(\delta_{i,1}) = \text{len}(\delta_{j,k}) \geq \text{len}(\alpha)/2$ ; 5) all the common paths shared by  $\varphi^i(\alpha)$  and  $\varphi^j(\alpha)$  have length  $\geq 1$ , except the terminal one ending at  $\varphi^j(v) = \varphi^i(v')$ , which can have length zero.*

*Proof.* Look at Figure 6 assuming  $\varphi^j(v) = \varphi^i(v')$  and  $\varphi^j(v') \neq \varphi^i(v)$ . By a similar argument as in the proof of the previous lemma, the minimality of  $\alpha$  implies that  $\varphi^i(\alpha)$  and  $\varphi^j(\alpha)$  cannot share an edge  $e$  like in the left side of the figure, while if  $\varphi^i(\alpha)$  and  $\varphi^j(\alpha)$  share an edge  $e$  like in the right side of the figure then  $\text{len}(\beta_{i,1}) = \text{len}(\beta_{j,2}) \geq \text{len}(\alpha)/2$ . This gives properties 1 and 4.

For property 2, it suffices to observe that, if  $e$  and  $e'$  are two common edges of  $\varphi^i(\alpha)$  and  $\varphi^j(\alpha)$ , then they occur in opposite orders along the two paths starting from  $\varphi^i(v)$  and  $\varphi^i(v')$  respectively. Indeed, in the contrary case, we would have  $\varphi^i(\alpha) = \beta_{i,1}e\beta_{i,2}e'\beta_{i,3}$  and  $\varphi^j(\alpha) = \beta_{j,1}\bar{e}\beta_{j,2}\bar{e}'\beta_{j,3}$  like in Figure 8, where  $\beta_i$ 's and the  $\beta_j$ 's are possibly empty and non-necessarily disjoint. Then, the minimality of  $\varphi^i(\alpha)$  would imply  $\text{len}(\beta_{i,2}) + 2 \leq \text{len}(\beta_{j,2})$ , while the minimality of  $\varphi^j(\alpha)$  would imply  $\text{len}(\beta_{j,2}) + 2 \leq \text{len}(\beta_{i,2})$ , which would be absurd.

At this point, property 3 and 5 immediately follow from the minimality of  $\varphi^i(\alpha)$  and  $\varphi^j(\alpha)$  and from the trivalency of the vertices respectively.  $\square$

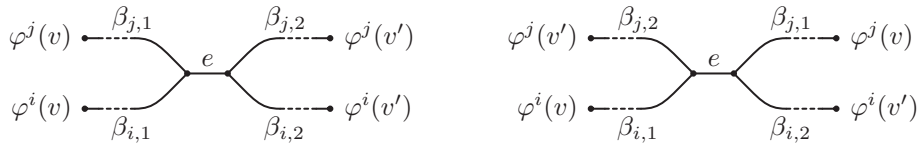


FIGURE 6. No common edge  $e$  between  $\varphi^i(\alpha)$  and  $\varphi^j(\alpha)$  with four distinct ends.

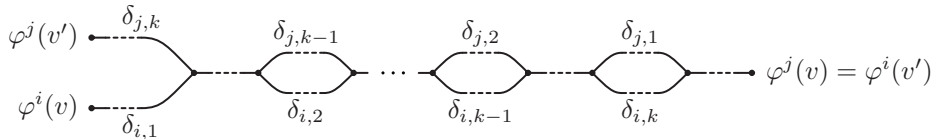


FIGURE 7.  $\varphi^i(\alpha) \cup \varphi^j(\alpha)$  for  $\varphi^j(v) = \varphi^i(v')$  and  $\varphi^j(v') \neq \varphi^i(v)$ .

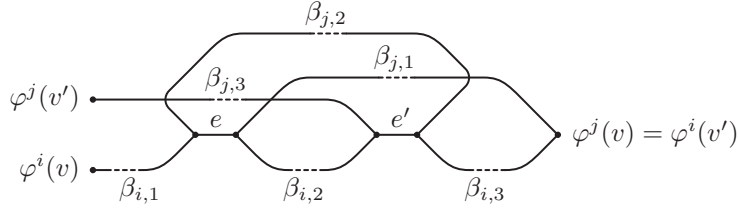


FIGURE 8. Common edges  $e$  and  $e'$  cannot occur in the same order along the paths  $\varphi^i(\alpha)$  and  $\varphi^j(\alpha)$  in Lemma 2.4.

**Lemma 2.5.** *Let  $\varphi : \Gamma \rightarrow \Gamma$ ,  $\alpha \subset \Gamma$ ,  $v$  and  $v'$ , be as in Lemma 2.3. If  $i \neq j$  with  $\varphi^j(v) = \varphi^i(v')$  and  $\varphi^j(v') = \varphi^i(v)$ , then the union  $\varphi^i(\alpha) \cup \varphi^j(\alpha)$  has the structure shown in Figure 9, where: 1)  $k \geq 0$ ; 2) the paths  $\delta_{i,1}, \dots, \delta_{i,k} \subset \varphi^i(\alpha)$  are disjoint from the paths  $\delta_{j,1}, \dots, \delta_{j,k} \subset \varphi^j(\alpha)$ ; 3)  $\text{len}(\delta_{i,h}) = \text{len}(\delta_{j,k-h+1}) \geq 1$  for every  $h = 1, \dots, k$ ; 4) all the common paths shared by  $\varphi^i(\alpha)$  and  $\varphi^j(\alpha)$  have length  $\geq 1$ , except both the terminal ones that can have length zero.*

Moreover, this may happen only when  $\text{ord}(\varphi)$  is even.

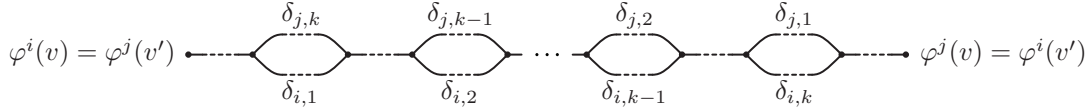


FIGURE 9.  $\varphi^i(\alpha) \cup \varphi^j(\alpha)$  for  $\varphi^i(v) = \varphi^j(v')$  and  $\varphi^j(v) = \varphi^i(v')$ .

*Proof.* The same arguments in the proof of Lemma 2.4 still work here, to give the first part of the statement. In fact, in that proof the assumption that  $\varphi^j(v') \neq \varphi^i(v)$  was only used to obtain  $k \geq 1$  and property 4 (in the statement of that lemma).

For the last sentence, we observe that the conditions  $\varphi^j(v) = \varphi^i(v')$  and  $\varphi^j(v') = \varphi^i(v)$  imply that  $\varphi^{j-i}$  swaps  $v$  and  $v'$ , which in turn implies that  $\text{ord}(\varphi^{j-i})$ , and hence  $\text{ord}(\varphi)$  as well, is even.  $\square$

**Lemma 2.6.** *Let  $\varphi : \Gamma \rightarrow \Gamma$ ,  $\alpha \subset \Gamma$ ,  $v$  and  $v'$ , be as in Lemma 2.3. If  $i \neq j$  with  $\varphi^i(v) = \varphi^j(v)$  and  $\varphi^i(v') = \varphi^j(v')$ , then the union  $\varphi^i(\alpha) \cup \varphi^j(\alpha)$  has the structure shown in Figure 10, where: 1)  $k \geq 0$ ; 2) the paths  $\delta_{i,1}, \dots, \delta_{i,k} \subset \varphi^i(\alpha)$  are disjoint from the paths  $\delta_{j,1}, \dots, \delta_{j,k} \subset \varphi^j(\alpha)$ ; 3)  $\text{len}(\delta_{i,h}) = \text{len}(\delta_{j,h}) \geq 1$  for every  $h = 1, \dots, k$ ; 4) all the common paths shared by  $\varphi^i(\alpha)$  and  $\varphi^j(\alpha)$  have length  $\geq 1$ , except both the terminal ones that can have length zero.*

Moreover, apart for the trivial case of  $\varphi^i(\alpha) = \varphi^j(\alpha)$ , this may happen only when  $\text{ord}(\varphi)$  is either even or an odd multiple of 3, being in the latter case  $\varphi^i(\alpha) = \delta_{i,1}$  and  $\varphi^j(\alpha) = \delta_{j,1}$ .

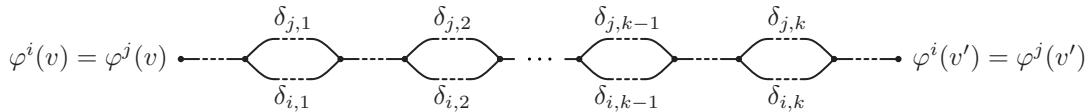


FIGURE 10.  $\varphi^i(\alpha) \cup \varphi^j(\alpha)$  for  $\varphi^i(v) = \varphi^j(v)$  and  $\varphi^i(v') = \varphi^j(v')$ .

*Proof.* The proof of the first part of the statement is completely analogous to that of the previous lemma, except for the orderings of the  $\delta_i$ 's along  $\varphi^i(\alpha)$  and of the  $\delta_j$ 's along  $\varphi^j(\alpha)$ , which now coincide.

To prove the second part, assume that  $\varphi^i(\alpha) \neq \varphi^j(\alpha)$  and  $\text{ord}(\varphi)$  is odd. Then, we have  $\varphi^i(\alpha) = \delta_{i,1}$  and  $\varphi^j(\alpha) = \delta_{j,1}$ . Otherwise,  $\varphi^{j-i}$  would fix some edge of  $\varphi^i(\alpha) \cap \varphi^j(\alpha)$  and  $\text{ord}(\varphi^{j-i})$  should be even by Lemma 2.2, in contrast with the oddness of  $\text{ord}(\varphi)$ . For the same reason  $\varphi^{j-i}$  cannot swap  $\varphi^i(\alpha)$  and  $\varphi^j(\alpha)$ , then it cyclically permutes the three paths  $\varphi^i(\alpha)$ ,  $\varphi^j(\alpha)$  and  $\varphi^{2j-i}(\alpha)$ , which meet each other only at their common ends  $\varphi^i(v)$  and  $\varphi^i(v')$ . This implies that  $\text{ord}(\varphi^{j-i})$ , and hence  $\text{ord}(\varphi)$  as well, is a multiple of 3.  $\square$

### 3. THE CASE OF ORDER $p^m$ WITH $p$ PRIME GREATER THAN 3

The aim of this section is to show that any automorphism  $\varphi \in \mathcal{A}_{g,b}$  with order  $\text{ord}(\varphi) = p^m$  for a prime  $p > 3$  is  $F$ -equivalent to the composition of two automorphisms of order 2.

Actually, we will prove this fact under the weaker assumption that  $n = \text{ord}(\varphi)$  is a multiple of neither 2 nor 3. In fact, all the arguments are only based on the following special properties, which hold for any automorphism  $\varphi : \Gamma \rightarrow \Gamma$  of a (possibly disconnected) uni/trivalent graph  $\Gamma$ , having such an order  $n$ :

- 1)  $\text{ord}_\varphi(v) = \text{ord}_\varphi(e) = n$  for any vertex  $v$  and edge  $e$  of  $\Gamma$ , thanks to Lemma 2.2;
- 2) the situations of Lemmas 2.5 and 2.6 cannot occur.

**Lemma 3.1.** *Let  $\varphi : \Gamma \rightarrow \Gamma$  be an automorphism of a (possibly disconnected) uni/trivalent graph  $\Gamma$ , whose order  $n = \text{ord}(\varphi)$  is a multiple of neither 2 nor 3. If  $\alpha \subset \Gamma$  is a (simple) path of minimal length among all the paths joining any two distinct  $\varphi$ -equivalent vertices, then  $\cup_i \varphi^i(\alpha)$  is a disjoint union  $C \sqcup \varphi(C) \sqcup \dots \sqcup \varphi^{n/\ell-1}(C)$  of  $n/\ell$  simple cycles (possibly a single one, for  $\ell = n$ ), each given by a concatenation of  $\ell$  images of  $\alpha$  with  $\ell$  a divisor of  $n$  greater than 1. In fact, there exists a positive integer  $s = tn/\ell < n$  with  $(t, \ell) = 1$ , such that the cycle  $C$  is given by  $\alpha \varphi^s(\alpha) \dots \varphi^{(\ell-1)s}(\alpha)$ . Moreover, up to  $F$ -equivalence we can assume  $\alpha$  to consist of a single edge  $a$  (see Figure 11, where  $a_i$  stands for  $\varphi^{is}(a)$  and a similar notation is adopted for the  $v_i$ 's as well).*

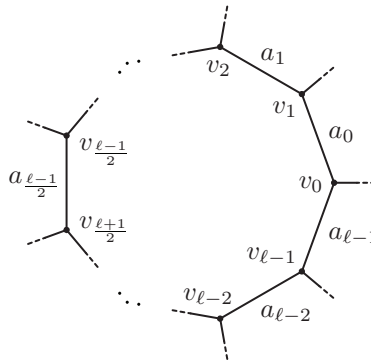


FIGURE 11. The form of the cycle  $C$  in Lemma 3.1.



*Proof.* Let  $\alpha$  be a path as in the statement, and  $e \neq e'$  be the terminal edges it joins. Then the ends of  $\alpha$  coincide with the unique trivalent ends  $v$  and  $v'$  of  $e$  and  $e'$  respectively. Notice that  $v \neq v'$ , being  $\text{ord}_\varphi(v) = \text{ord}_\varphi(e) = n$  by Lemma 2.2.

For any  $i \neq j \pmod n$ , by arguing as in the proof of Lemma 3.1, we can prove that  $\varphi^i(\alpha)$  and  $\varphi^j(\alpha)$  can meet only as in Lemma 2.4, and that their common end  $\varphi^j(v) = \varphi^i(v')$  cannot be shared by any other  $\varphi^h(\alpha)$ . The same Lemma 2.4 also tells us that  $\text{len}(\delta_{i,1}) = \text{len}(\delta_{j,k}) \geq \text{len}(\alpha)/2$  (cf. Figure 7). But here the equality cannot occur, otherwise  $\varphi^{j-i}$  would fix the common end of  $\delta_{i,1}$  and  $\delta_{j,k}$ , and it would cyclically permute the three edges exiting from it, in contrast with the assumption that  $n$  is not a multiple of 3.

Then, we can conclude straightforwardly that  $\cup_i \varphi^i(\alpha)$  consists of the disjoint union  $C \sqcup \varphi(C) \sqcup \dots \sqcup \varphi^{n/\ell-1}(C)$ , with  $s < n$  uniquely determined by  $\varphi^s(v) = v'$ ,  $\ell$  the smallest positive integer such that  $\varphi^{\ell s}(v) = v$ ,  $t = s\ell/n$  (cf. proof of Lemma 3.1), and  $C = \alpha \varphi^s(\alpha) \dots \varphi^{(\ell-1)s}(\alpha)$  having the form depicted in Figure 14. Here, and in the next figure as well,  $\alpha_i$  stands for the path  $\varphi^{is}(\alpha)$ , which is the part of  $C$  in the corresponding circular sector, and each arc represents a path of edges.

In the very special case where the paths  $\alpha_i$  consist of single edges, they can only share their common ends. Hence, the configuration  $C$  in Figure 14 coincides with that in the left side of Figure 13, and we are done.

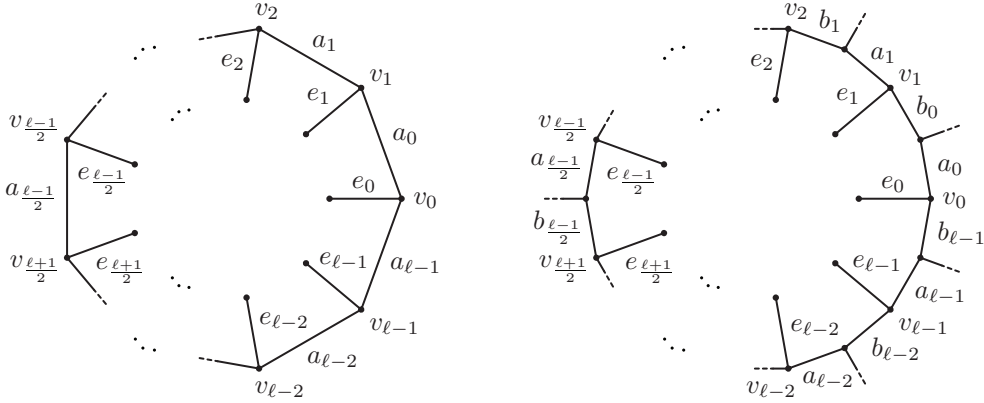


FIGURE 13. The two possible forms of the cycle  $C$  in Lemma 3.2.

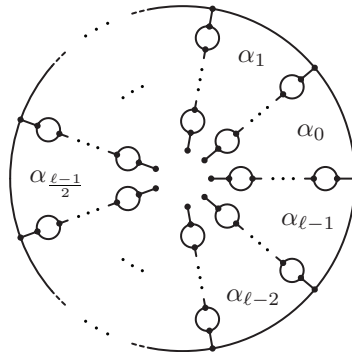


FIGURE 14. The starting configuration  $C$  in the proof of Lemma 3.2.

Otherwise, by a sequence of  $\varphi$ -invariant slidings (cf. Remark 1.5) on the edges attached to  $C$  and to its copies, as indicated in Figure 15 for three of such edges, we can reduce all the arcs in Figure 14 to single edges, possibly except the ones forming the big circle. In the same way, when the first edge of  $\alpha_i$  coincides with the last edge of  $\alpha_{i-1 \bmod \ell}$ , we also fuse it with the edge  $e_i = \varphi^{is}(e)$ .

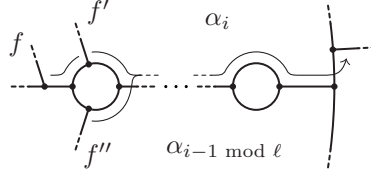


FIGURE 15. Simplifying the starting configuration.

Then, we can eliminate all the resulting bigons, by a  $\varphi$ -invariant  $F$ -move, which acts on  $C$  by simultaneous elementary  $F$ -moves on the edges between any two consecutive of them and between the most external ones and the big circle, as shown in Figure 16.

Finally, to get the configuration in the right side of Figure 13, it remains to reduce  $\alpha$  to a chain of only two edges  $a$  and  $b$ . This can be done proceeding by induction, like in the last part of the proof of Lemma 3.1. In Figure 17 we see how to reduce  $\text{len}(\alpha)$  whenever this is greater than 2. Here,  $a_i = \varphi^{is}(a)$  and  $f_i = \varphi^{is}(f)$  are respectively the images in  $\alpha_i = \varphi^{is}(\alpha)$  of the first edge  $a$  of  $\alpha$  and of the edge  $f$  attached at the second intermediate vertex of  $\alpha$  but not belonging to  $\alpha$ . The desired reduction is achieved through an invariant  $F$ -move, which simultaneously slides all the  $f_i$ , in such a way that their ends pass from the  $\alpha_i$ 's to the first edges attached to them. Notice that, contrary to what we did in the proof of Lemma 3.1, we cannot further reduce  $\alpha$  to a single edge by a slide letting even the last intermediate point of  $\alpha$  pass to  $e$ .  $\square$

**Proposition 3.3.** *Let  $\varphi \in \mathcal{A}_{g,b}$  be a non-trivial automorphism, whose order  $n = \text{ord}(\varphi) > 1$  is a multiple of neither 2 nor 3. Then,  $\varphi$  is  $F$ -equivalent to the composition  $\tau \circ \sigma$  of two automorphisms  $\sigma, \tau \in \mathcal{A}_{g,b}$  such that  $\text{ord}(\sigma) = \text{ord}(\tau) = 2$ . Moreover, both  $\sigma$  and  $\tau$  can be assumed to fix an edge and reverse an invariant edge.*

*Proof.* Let  $\varphi : \Gamma \rightarrow \Gamma$  any automorphism in  $\mathcal{A}_{n,g}$  as in the statement. We will construct by recursion a sequence  $\varphi_i : \Gamma_i \rightarrow \Gamma_i$  of automorphisms in  $\mathcal{A}_{g,b}$  and a sequence of subgraphs  $\Lambda_i \subset \Gamma_i$  with  $i = 0, \dots, r$  and  $r \geq 2$ , having the following properties:

- 1)  $\varphi_0 = \varphi$  (hence  $\Gamma_0 = \Gamma$ ), and  $\varphi_i$  is  $F$ -equivalent to  $\varphi_{i-1}$  for every  $1 \leq i \leq r$ ;
- 2)  $\emptyset = \Lambda_0 \subsetneq \Lambda_1 \subsetneq \dots \subsetneq \Lambda_r = \Gamma_r$ , and  $\Delta_i = \text{Cl}(\Gamma_i - \Lambda_i)$  is a uni/trivalent graph whose intersection with  $\Lambda_i$  is a single  $\varphi_i$ -orbit  $U_i \subset \Delta_i$  of its free ends for every  $1 \leq i \leq r-1$ ;
- 3)  $\Lambda_i$  is  $\varphi_i$ -invariant and  $\varphi_i|_{\Lambda_{i-1}} = \varphi_{i-1}|_{\Lambda_{i-1}}$  for every  $1 \leq i \leq r$ ;
- 4)  $\varphi_i|_{\Lambda_i} : \Lambda_i \rightarrow \Lambda_i$  is the composition  $\tau_i \circ \sigma_i$  of two automorphisms  $\sigma_i : \Lambda_i \rightarrow \Lambda_i$  and  $\tau_i : \Lambda_i \rightarrow \Lambda_i$  such that  $\text{ord}(\sigma_i) = \text{ord}(\tau_i) = 2$ , for every  $1 \leq i \leq r$ ; moreover,  $\sigma_i$  fixes exactly one vertex  $u_i$  in  $U_i$  (hence  $\tau_i$  fixes  $\varphi_i^{(n+1)/2}(u_i) \in U_i$ ) for every  $1 \leq i \leq r-1$ , and both  $\sigma_i$  and  $\tau_i$  fix an edge and reverse an invariant edge for every  $2 \leq i \leq r$ .

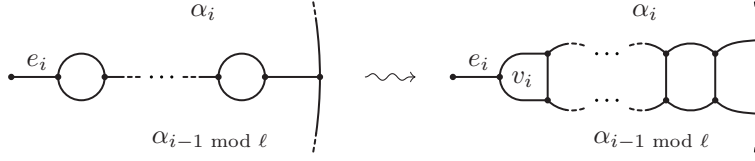


FIGURE 16. Eliminating the bigons by simultaneous edge  $F$ -moves.

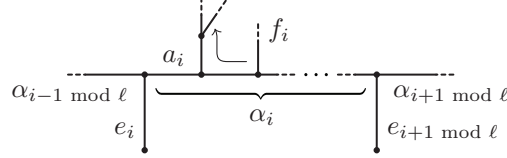


FIGURE 17. Reducing  $\text{len}(\alpha)$  in Lemma 3.2.

This will prove the proposition, since  $\sigma = \sigma_r$  and  $\tau = \tau_r$  satisfy all the required conditions, being  $\tau \circ \sigma = \tau_r \circ \sigma_r = \varphi_r|_{\Lambda_r} = \varphi_r$   $F$ -equivalent to  $\varphi$  in  $\mathcal{A}_{g,b}$ .

To start the construction, we apply Lemma 3.1 to the automorphism  $\varphi$  and a minimal path  $\alpha \subset \Gamma$  as in that statement (this exists since  $\Gamma$  is connected but not reduced to a single vertex and  $\varphi$  is non-trivial). As a result, we get an automorphism  $\varphi_1 : \Gamma_1 \rightarrow \Gamma_1$  in  $\mathcal{A}_{g,b}$ , which is  $F$ -equivalent to  $\varphi_0 = \varphi$  and such that the minimal path corresponding to  $\alpha$  in  $\Gamma_1$  consists of only one (internal) edge  $a \subset \Gamma_1$ . Then, we put  $\Lambda_1 = \cup_j \varphi_1^j(a) \subset \Gamma_1$ , which is  $\varphi_1$ -invariant by definition. Lemma 3.1 tells us that there exist a divisor  $\ell > 1$  of  $n$  and an integer  $0 < t < \ell$  with  $(t, \ell) = 1$ , such that  $\Lambda_1$  consists of the disjoint union  $C \sqcup \varphi_1(C) \sqcup \dots \sqcup \varphi_1^{n/\ell-1}(C)$  of  $n/\ell$  simple cycles (possibly a single one, for  $\ell = n$ ), where  $C = a \varphi_1^s(a) \dots \varphi_1^{(\ell-1)s}(a)$  with  $s = tn/\ell$ .

Let  $v$  be the first end of the edge  $a$ , so that  $\varphi_1^s(v)$  is the second one. Then, the set of vertices of  $\Lambda_1$  is the  $\varphi_1$ -orbit  $V = \{v, \varphi_1(v), \dots, \varphi_1^{n-1}(v)\}$ , and any automorphism on  $\Lambda_1$  is uniquely determined by its action on  $V$ , since  $n$  is odd and hence  $\ell > 2$ . Therefore, we can first define the two involutions  $\sigma_1$  and  $\tau_1$  on  $V$  in such a way that  $\tau_1 \circ \sigma_1$  coincides with the restriction of  $\varphi_1$  to  $V$ , then check that they preserve adjacency of vertices in  $\Lambda_1$  and hence give well-defined automorphisms of  $\Lambda_1$ .

We define  $\sigma_1$  and  $\tau_1$  on  $V$  by putting  $\sigma_1(\varphi_1^j(v)) = \varphi_1^{-j}(v)$  and  $\tau_1(\varphi_1^j(v)) = \varphi_1^{1-j}(v)$ , for  $j = 0, \dots, n-1$ . Clearly, these are involutions and their composition  $\tau_1 \circ \sigma_1$  acts on  $V$  as  $\varphi_1$ . Concerning the preservation of adjacency, it suffices to observe that two elements  $\varphi_1^j(v)$  and  $\varphi_1^k(v)$  of  $V$  are adjacent vertices in  $\Lambda_1$  if and only if  $|j - k| = s \pmod n$ , and both  $\sigma_1$  and  $\tau_1$  preserve this condition.

Finally, we note that  $\Delta_1 = \text{Cl}(\Gamma_1 - \Lambda_1)$  is a uni/trivalent graph and  $\Delta_1 \cap \Lambda_1$  is given by the  $\varphi_1$ -orbit  $U_1 = V$  of free ends of  $\Delta_1$ , as required in point 2 above. Moreover,  $\sigma_1$  fixes only the vertex  $u_1 = v$  of  $U_1$  and reverses the edge  $\varphi_1^{(n+1)/2}(a)$ , while  $\tau_1$  reverses the edge  $a$ , as required in point 4 above.

Now, to realize the recursive step of the construction, assume we are given  $\varphi_{i-1} : \Gamma_{i-1} \rightarrow \Gamma_{i-1}$ ,  $u_{i-1} \in U_{i-1} \subset \Lambda_{i-1} \subsetneq \Gamma_{i-1}$  and  $\sigma_{i-1}, \tau_{i-1} : \Lambda_{i-1} \rightarrow \Lambda_{i-1}$  satisfying all the properties stated in the above points 1 to 4 for  $i-1 < r$ . Lemmas 2.1 and 2.2 ensure that  $\text{ord}(\varphi_{i-1}) = n$  and that the cardinality of the  $\varphi_{i-1}$ -orbit  $U_{i-1}$  is equal to  $n$  as well. Hence, also the restriction of  $\varphi_{i-1}$  to the uni/trivalent

graph  $\Delta_{i-1} = \text{Cl}(\Gamma_{i-1} - \Lambda_{i-1})$  is an automorphism of the same order  $n$ . Furthermore, the terminal edges of  $\Delta_{i-1}$  ending at vertices in  $U_{i-1}$  are all distinct. In fact, if two of them would coincide, then their  $\varphi$ -order would be even, which contradicts the hypothesis that  $n$  is odd.

If  $\Delta_{i-1}$  does not contain any path joining different terminal edges ending at vertices in  $U_{i-1}$ , then it consists of  $n$  components each containing a single vertex in  $U_{i-1}$ . Denoting by  $C$  the component of  $\Delta_{i-1}$  containing that vertex  $u_{i-1}$ , we have the component decomposition  $\Delta_{i-1} = C \sqcup \varphi_{i-1}(C) \sqcup \dots \sqcup \varphi_{i-1}^{n-1}(C)$ , with  $\varphi_{i-1}$  cyclically permuting the components. In this case, we put  $\varphi_i = \varphi_{i-1}$  and  $\Lambda_i = \Gamma_i = \Gamma_{i-1}$ . Moreover, we define  $\sigma_i$  and  $\tau_i$  as the unique automorphisms of  $\Gamma_i$  extending  $\sigma_{i-1}$  and  $\tau_{i-1}$ , in such a way that they act on each  $\varphi_{i-1}^j(C)$  with  $j = 0, \dots, n-1$ , as  $\varphi_{i-1}^{-2j}$  and  $\varphi_{i-1}^{1-2j}$  respectively. A straightforward verification shows that such  $\varphi_i$ ,  $\Lambda_i$ ,  $\sigma_i$  and  $\tau_i$  satisfy all the properties stated in the above points 1 to 4 for the case when  $i = r$ , which means that this step terminates the recursion.

Otherwise, if a path  $\alpha \subset \Delta_{i-1}$  exists joining different terminal edges of  $\Delta_{i-1}$  ending at vertices in  $U_{i-1}$ , then we can choose it to be minimal (hence simple) and apply Lemma 3.2 to the restriction of  $\varphi_{i-1}$  to  $\Delta_{i-1}$  and to such a minimal  $\alpha$ , in order to get the structure described in that statement for  $\cup_j \varphi_{i-1}^j(\alpha) \subset \Delta_{i-1}$  up to  $F$ -equivalence. Such an  $F$ -equivalence only involves internal edges of  $\Delta_{i-1}$ , hence it does not change the set of free ends  $U_{i-1} \subset \Delta_{i-1}$  and the restriction of  $\varphi_{i-1}|_{\Delta_{i-1}}$ . Therefore, it can be extended to an  $F$ -equivalence of the whole  $\varphi_{i-1}$  on  $\Gamma_{i-1}$ , which leaves  $\Lambda_{i-1}$  and the restriction  $\varphi_{i-1}|_{\Lambda_{i-1}}$  unchanged.

As a result we get a new uni/trivalent graph  $\Gamma_i$ , such that  $\Lambda_{i-1} \subset \Gamma_i$  and a new automorphism  $\varphi_i : \Gamma_i \rightarrow \Gamma_i$ , which is  $F$ -equivalent to  $\varphi_{i-1}$  and coincides with  $\varphi_{i-1}$  on  $\Lambda_{i-1}$ . We also get a new minimal path  $\alpha \subset \text{Cl}(\Gamma_i - \Lambda_{i-1})$ , which join different  $\varphi_i$ -equivalent terminal edges of  $\text{Cl}(\Gamma_i - \Lambda_{i-1})$  ending at vertices in  $U_{i-1}$ , such that  $\cup_j \varphi_i^j(\alpha)$  itself (no more up to  $F$ -equivalence) has the structure described in Lemma 3.2. Let  $e \subset \Gamma_i$  be the terminal edge of  $\text{Cl}(\Gamma_i - \Lambda_{i-1})$  ending at  $u_{i-1}$ , and  $v \in \Gamma_i$  be the other end of  $e$ , which is a trivalent vertex of  $\text{Cl}(\Gamma_i - \Lambda_{i-1})$ . Up to  $\varphi_i$ -equivalence, we can assume that  $\alpha \subset \Gamma_i$  starts at  $v$  and ends at  $\varphi_i^s(v)$  with  $0 < s < n$ . According to Lemma 3.2, we can also assume that  $\alpha$  consists of either one edge  $a$  or two edges  $a$  and  $b$  sharing the vertex  $u$ , while  $\cup_j \varphi_i^j(\alpha)$  consists of the disjoint union  $C \sqcup \varphi_i(C) \sqcup \dots \sqcup \varphi_i^{n/\ell-1}(C)$  of  $n/\ell$  cycles (possibly a single one, for  $\ell = n$ ), where  $C = \alpha \varphi_i^s(\alpha) \dots \varphi_i^{(\ell-1)s}(\alpha)$  for some  $\ell > 1$  such that  $s\ell = tn$  with  $(t, \ell) = 1$ .

We put  $\Lambda_i = \Lambda_{i-1} \cup_j \varphi_i^j(e) \cup_j \varphi_i^j(\alpha)$  and define  $\sigma_i$  and  $\tau_i$  to be the unique automorphisms of  $\Lambda_i$  extending  $\sigma_{i-1}$  and  $\tau_{i-1}$  respectively. To see that such extensions exist, we first define  $\sigma_i(\varphi_i^j)(v) = \varphi_i^{-j}(v)$  and  $\tau_i(\varphi_i^j)(v) = \varphi_i^{1-j}(v)$ , and also  $\sigma_i(\varphi_i^j)(u) = \varphi_i^{-j}(u)$  and  $\tau_i(\varphi_i^j)(u) = \varphi_i^{1-j}(u)$  in the case when  $\alpha = ab$ , for every  $j = 0, \dots, n-1$ . Then, essentially by the same argument used above for  $i = 1$ , we check that these definitions, together with  $\sigma_i(\varphi_i^j)(u_{i-1}) = \varphi_i^{-j}(u_{i-1})$  and  $\tau_i(\varphi_i^j)(u_{i-1}) = \varphi_i^{1-j}(u_{i-1})$  for every  $i = 0, \dots, n$ , preserve adjacency of vertices in  $\Lambda_i$ , hence they determine automorphisms  $\sigma_i$  and  $\tau_i$  of the graph  $\Lambda_i$  extending  $\sigma_{i-1}$  and  $\tau_{i-1}$  respectively. Notice that  $\tau_i \circ \sigma_i$  trivially coincides with  $\varphi_i|_{\Lambda_i}$ . Moreover,  $\sigma_i$  fixes the edge  $e$  and reverses the edge  $\varphi_i^{(n+1)/2}(a)$ , while  $\tau_i$  fixes the edge  $\varphi_i^{(n+1)/2}(e)$  and reverses the edge  $a$ . Finally,  $u_i = \varphi_i^{(n+1)/2}(u)$  is the unique vertex in its  $\varphi_i$ -orbit  $U_i = \{\varphi_i^j(u) \mid j = 0, \dots, n\}$  fixed by  $\sigma_i$ .

At this point, if  $\alpha$  consists of the only edge  $a$ , hence  $C$  has the structure depicted in the left side of Figure 13, then  $\Lambda_i = \Gamma_i$  by the connectedness of  $\Gamma_i$  and the recursion terminates with  $r = i$ . Otherwise, if  $\alpha$  consists of the two edges  $a$  and  $b$ , hence  $C$  has the structure depicted in the right side of Figure 13, then we put  $\Delta_i = \text{Cl}(\Gamma_i - \Lambda_i)$  and observe that this is a uni/trivalent graph such that  $\Delta_i \cap \Lambda_i = U_i$ . This concludes the recursive step.  $\square$

#### 4. THE CASE OF ORDER $3^m$

In this section, we want to prove that any automorphism  $\varphi \in \mathcal{A}_{g,b}$  with order  $\text{ord}(\varphi) = 3^m$  is  $F$ -equivalent to the composition of two automorphisms of order 2.

Like in the previous section, a weaker assumption will suffice to get the same conclusion. Namely, we will assume that  $n = \text{ord}(\varphi)$  is an odd multiple of 3. The only properties we will need, for any automorphism  $\varphi : \Gamma \rightarrow \Gamma$  of a (possibly disconnected) uni/trivalent graph  $\Gamma$  with such an order  $n$ , are the following:

- 1)  $\text{ord}_\varphi(e) = n$  for any edge  $e$  of  $\Gamma$ , while  $\text{ord}_\varphi(v)$  is either  $n$  or  $n/3$  for any vertex  $v$  of  $\Gamma$ , thanks to Lemma 2.2;
- 2) the situation of Lemmas 2.5 cannot occur, while that of Lemma 2.6 can only occur with the two paths  $\varphi^i(\alpha)$  and  $\varphi^j(\alpha)$  sharing no edge.

The whole argument is essentially the same as in the previous section, except for some more cases occurring here. Hence, in both statements and proofs we will just concern with those extra cases, while referring for the others to the analogous statements and proofs in the previous section.

**Lemma 4.1.** *Let  $\varphi : \Gamma \rightarrow \Gamma$  be an automorphism of a (possibly disconnected) uni/trivalent graph  $\Gamma$ , whose order  $n = \text{ord}(\varphi)$  is an odd multiple of 3. If  $\alpha \subset \Gamma$  is a (simple) path of minimal length among all paths joining any two distinct  $\varphi$ -equivalent vertices, then  $\cup_i \varphi^i(\alpha)$  is one of the following:*

- 1) a disjoint union  $C \sqcup \varphi(C) \sqcup \dots \sqcup \varphi^{n/\ell-1}(C)$  of  $n/\ell$  simple cycles (possibly a single one, for  $\ell = n$ ), each given by a concatenation of images of  $\alpha$ , having the same structure as described in Lemma 3.1; in this case, up to  $F$ -equivalence we can assume  $\alpha$  to consist of a single edge  $a$  (see Figure 11, where  $a_i$  stands for  $\varphi^{is}(a)$  and a similar notation is adopted for the vertices  $v_i$  as well);
- 2) a disjoint union  $T \sqcup \varphi(T) \sqcup \dots \sqcup \varphi^{n/3-1}(T)$  of  $n/3$  tripods (possibly a single one, for  $n = 3$ ), where  $T = \alpha \cup \varphi^{n/3}(\alpha) \cup \varphi^{2n/3}(\alpha)$  and  $\alpha$  is the concatenation of two edges  $a$  and  $b$ , such that  $b = \varphi^{n/3}(\bar{a})$ ,  $\varphi^{n/3}(b) = \varphi^{2n/3}(\bar{a})$  and  $\varphi^{2n/3}(b) = \bar{a}$  (see Figure 18, where  $a_i$  stands for  $\varphi^{in/3}(a)$  and a similar notation is adopted for the vertices  $v_i$ , which can be either all univalent as on the left side or all trivalent as on the right side).

*Proof.* Let  $\alpha \subset \Gamma$  be an arbitrary path of minimal length among all paths joining any two distinct  $\varphi$ -equivalent vertices and let  $v \neq v'$  be its ends. For any  $i \neq j \pmod n$ , let us consider the two paths  $\varphi^i(\alpha)$  and  $\varphi^j(\alpha)$ , and the four different situations described in Lemmas 2.3 to 2.6, which cover all the possibilities, thanks to the  $\varphi$ -equivalence of  $v$  and  $v'$ .

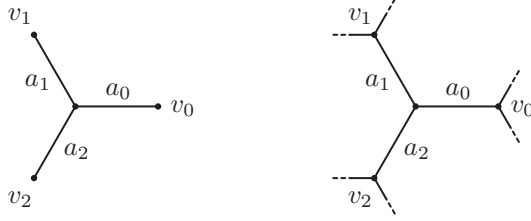


FIGURE 18. The form of the tripod  $T$  in Lemma 4.1.

The situation of Lemma 2.5 cannot occur, while that of Lemma 2.6 could possibly occur only with  $\varphi^i(\alpha)$  and  $\varphi^j(\alpha)$  sharing no edge, but the argument below shows this is actually impossible.

Assume by contradiction that  $\varphi^i(\alpha)$  and  $\varphi^j(\alpha)$  are as in Lemma 2.6, hence they meet at  $\varphi^i(v) = \varphi^j(v)$  and  $\varphi^i(v') = \varphi^j(v')$ . Let us denote by  $a$  the first edge of  $\alpha$  starting from  $v$  and by  $u$  the other end of  $a$ . Then, the vertices  $\varphi^i(u)$  and  $\varphi^j(u)$  are distinct since  $\varphi^i(\alpha)$  and  $\varphi^j(\alpha)$  do not share any edge. Being  $n$  an odd multiple of 3, thanks to Lemma 2.2 we have  $\text{ord}_\varphi(v) = n/3$  and  $\text{ord}_\varphi(u) = n$ .

As a consequence, we get  $\text{len}(\alpha) = 2$ . In fact, the concatenation  $\varphi^i(\bar{a})\varphi^j(a)$  is simple path of length 2 between the two distinct  $\varphi$ -equivalent vertices  $\varphi^i(u)$  and  $\varphi^j(u)$ . Hence, by the global minimality of  $\alpha$  we have  $\text{len}(\alpha) \leq 2$ . On the other hand,  $\alpha$  cannot be reduced to the single edge  $a$ , otherwise  $v$  and  $u$  should be  $\varphi$ -equivalent, in contrast with  $\text{ord}_\varphi(u) \neq \text{ord}_\varphi(v)$ .

Now, let  $s$  be an integer such that  $v' = \varphi^s(v)$ . Then,  $\varphi^s(u)$  coincides with one of  $u$ ,  $\varphi^{n/3}(u)$  and  $\varphi^{2n/3}(u)$ , since these are the three vertices adjacent to  $v'$ . In any case, we can conclude that  $s$  is a multiple of  $n/3$ , which is in contrast with  $\text{ord}_\varphi(v) = n/3$  and  $v' \neq v$ . This proves that the situation of Lemma 2.6 cannot occur.

At this point, we have that  $\varphi^i(\alpha)$  and  $\varphi^j(\alpha)$  can only meet as in Lemma 2.4. Let  $\varphi^i(u)$  with  $u \in \alpha$  be the common end of  $\delta_{i,1}$  and  $\delta_{j,k}$ . As in the proof of Lemma 3.1, we can see that either  $\varphi^j(u) = \varphi^j(v')$  or  $\varphi^j(u) = \varphi^i(u)$ , due to the global minimality of  $\alpha$ . But differently from that proof, here the latter equality can actually hold when  $u$  has order  $n/3$  and  $j = i \pmod{n/3}$ .

If  $\varphi^j(u) = \varphi^i(u)$ , then the global minimality of  $\alpha$  implies that both  $\delta_{i,1}$  and  $\delta_{j,k}$  have length 1. Therefore,  $\alpha$  is the concatenation of two edges  $a$  and  $b$  sharing the vertex  $u$ , and hence  $T = \alpha \cup \varphi^{n/3}(\alpha) \cup \varphi^{2n/3}(\alpha)$  has the form described in point 2 of the statement. Of course, this means that  $\cup_i \varphi^i(\alpha) = T \cup \varphi(T) \cup \dots \cup \varphi^{n/3-1}(T)$ . Moreover, any two different copies of  $T$  in this union are disjoint, otherwise they should share a free end, and there would be two paths  $\varphi$ -equivalent to  $\alpha$  exiting from that common vertex like in Lemma 2.6.

Notice that the last argument shows that, if  $\varphi^j(u) = \varphi^i(u)$  for some  $i$  and  $j$ , then the same holds for any  $i$  and  $j$  such that  $\varphi^i(\alpha)$  and  $\varphi^j(\alpha)$  are not disjoint.

So, we are left with the case when  $\varphi^j(u) = \varphi^j(v')$  for any two non-disjoint paths  $\varphi^i(\alpha)$  and  $\varphi^j(\alpha)$ . In this case, the same argument exploited in the proof of Lemma 3.1, allows us to conclude that the situation is the one described in point 1 of the statement.  $\square$

**Lemma 4.2.** *Let  $\varphi : \Gamma \rightarrow \Gamma$  be an automorphism of a (possibly disconnected) uni/trivalent graph  $\Gamma$ , whose order  $n = \text{ord}(\varphi)$  is an odd multiple of 3. If  $\alpha \subset \Gamma$  is a (simple) path, possibly degenerate to a single vertex, of minimal length among*

all paths joining any two distinct terminal edges in a given  $\varphi$ -orbit, then up to  $F$ -equivalence we can assume  $\cup_i \varphi^i(\alpha)$  to be one of the following:

- 1) a disjoint union  $C \sqcup \varphi(C) \sqcup \dots \sqcup \varphi^{n/\ell-1}(C)$  of  $n/\ell$  simple cycles (possibly a single one, for  $\ell = n$ ), each given by a concatenation of images of  $\alpha$ , having the same structure as described in Lemma 3.2, with  $\alpha$  consisting of either one edge  $a$  or two edges  $a$  and  $b$  (see Figure 13, where  $a_i$  and  $b_i$  stand for  $\varphi^{is}(a)$  and  $\varphi^{is}(b)$  respectively, and a similar notation is adopted for the terminal edges  $e_i$ 's and the vertices  $v_i$ 's as well);
- 2) a disjoint union  $T \sqcup \varphi(T) \sqcup \dots \sqcup \varphi^{n/3-1}(T)$  of  $n/3$  tripods (possibly a single one, for  $n = 3$ ), where  $T = \alpha \cup \varphi^{n/3}(\alpha) \cup \varphi^{2n/3}(\alpha)$ , with  $\alpha$  being the concatenation of two edges  $a$  and  $b$ , such that  $b = \varphi^{n/3}(\bar{a})$ ,  $\varphi^{n/3}(b) = \varphi^{2n/3}(\bar{a})$  and  $\varphi^{2n/3}(b) = \bar{a}$  (see left side of Figure 19, where  $a_i$  and  $e_i$  stand for  $\varphi^{in/3}(a)$  and  $\varphi^{in/3}(e)$  respectively);
- 3) a set of  $n/3$  trivalent vertices, when  $\alpha$  reduces to a single vertex, that is the terminal edges it joins share a common trivalent end; in this case the whole graph  $\Gamma$  consists of a disjoint union  $\widehat{T} \sqcup \varphi(\widehat{T}) \sqcup \dots \sqcup \varphi^{n/3-1}(\widehat{T})$  of  $n/3$  tripods, where  $\widehat{T} = e \cup \varphi^{n/3}(e) \cup \varphi^{2n/3}(e)$  with  $\alpha$  reduced to the trivalent vertex of  $\widehat{T}$  (see right side of Figure 19, where  $e_i$  stands for the terminal edge  $\varphi^{in/3}(e)$ ).

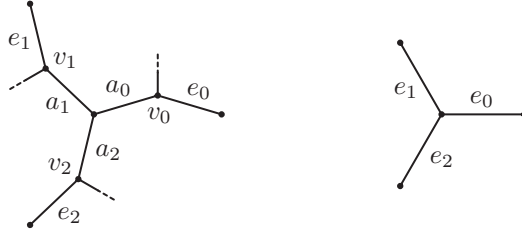


FIGURE 19. The form of the tripods  $T$  (left) and  $\widehat{T}$  (right) in Lemma 4.2.

*Proof.* Let  $\alpha$  be a path as in the statement and let  $e \neq e'$  be the terminal edges it joins. Then the ends of  $\alpha$  coincide with the unique trivalent ends  $v$  and  $v'$  of  $e$  and  $e'$  respectively.

If  $v = v'$ , that is  $\text{len}(\alpha) = 0$ , then  $\text{ord}_\varphi(v) = n/3$  according to Lemma 2.2 and  $e'$  is either  $\varphi^{n/3}(e)$  or  $\varphi^{2n/3}(e)$ . In any case,  $\varphi^{n/3}$  cyclically permutes the three edges of the tripod  $\widehat{T} = e \cup \varphi^{n/3}(e) \cup \varphi^{2n/3}(e)$ , and we have the situation described in point 3 of the statement.

If  $v \neq v'$ , that is  $\text{len}(\alpha) \geq 1$ , we consider any two distinct paths  $\varphi^i(\alpha)$  and  $\varphi^j(\alpha)$  with  $i \neq j \pmod n$  and look once again at the four possible situations described in Lemmas 2.3 to 2.6.

As in the proof of Lemma 4.1, the situations of Lemmas 2.5 and 2.6 cannot occur. But here the argument to exclude the latter is different. Namely, if  $\varphi^i(\alpha)$  and  $\varphi^j(\alpha)$  were as in Lemma 2.6, then  $\varphi^{j-i}$  should swap the first edges of them starting from  $\varphi^i(v) = \varphi^j(v)$ , being  $\varphi^i(e) = \varphi^j(e)$  the third edge at that vertex, and this would be in contrast with the oddness of  $n$ .

Therefore,  $\varphi^i(\alpha)$  and  $\varphi^j(\alpha)$  can meet only as in Lemma 2.4, and their common end  $\varphi^j(v) = \varphi^i(v')$  cannot be shared by any other  $\varphi^h(\alpha)$  (cf. proof of Lemma 3.1). Then, consider the two subpaths  $\delta_{i,1} \subset \varphi^i(\alpha)$  and  $\delta_{j,k} \subset \varphi^j(\alpha)$  in Figure 7.

If  $\text{len}(\delta_{i,1}) = \text{len}(\delta_{j,k}) = \text{len}(\alpha)/2$ , then for the middle vertex  $u$  of  $\alpha$  we have that  $\varphi^i(u) = \varphi^j(u)$  is the common end of  $\delta_{i,1}$  and  $\delta_{j,k}$ . By the same argument as above, this implies that  $\text{ord}_\varphi(u) = n/3$  and  $\varphi^{n/3}$  cyclically permutes the three paths  $\alpha$ ,  $\varphi^{n/3}(\alpha)$  and  $\varphi^{2n/3}(\alpha)$ . Moreover, such three paths are disjoint from any other path  $\varphi^h(\alpha)$ , hence we can conclude that  $\cup_i \varphi^i(\alpha)$  is a disjoint union  $T \sqcup \varphi(T) \sqcup \dots \sqcup \varphi^{n/3-1}(T)$ , with  $T$  as in the left side of Figure 20. Here,  $\alpha_i$  stands for the path  $\varphi^{in/3}(\alpha)$ , which is the part of  $T$  in the corresponding circular sector, and any arc except the terminal ones represent a path of edges. Analogously,  $v_i$  and  $e_i$  stand for  $\varphi^{in/3}(v)$  and  $\varphi^{in/3}(e)$  respectively.

We can reduce any arc in this starting configuration  $T$  to a single edge, by performing  $\varphi$ -invariant slidings as in the proof of Lemma 3.2 (cf. Figure 15), in order to push all the edges attached along the  $\alpha_i^s$  so that they become attached to the last one exiting from  $v_i$  (and different from  $e_i$ ). Then, the obvious  $\varphi$ -invariant  $F$ -moves allow us to change the resulting  $T$  into the form depicted on the right side of Figure 20, with each  $\alpha_i$  being now the shortest path between  $v_i$  and  $v_{i+1 \bmod 3}$ . Finally, we can get the configuration described in point 2 of the statement and shown in the left side of Figure 19, by further  $\varphi$ -invariant slidings of the new edges attached along the actual  $\alpha_i$  as above.

Otherwise, if  $\text{len}(\delta_{i,1}) = \text{len}(\delta_{j,k}) > \text{len}(\alpha)/2$ , then the same argument as in the proof of Lemma 3.2 still works here, to put  $\cup_i \varphi^i(\alpha)$  in the form stated in point 1 of the statement.  $\square$

**Proposition 4.3.** *Let  $\varphi \in \mathcal{A}_{g,b}$  be a non-trivial automorphism, whose order  $n = \text{ord}(\varphi) > 1$  is an odd multiple of 3. Then,  $\varphi$  is  $F$ -equivalent to the composition  $\tau \circ \sigma$  of two automorphisms  $\sigma, \tau \in \mathcal{A}_{g,b}$  such that  $\text{ord}(\sigma) = \text{ord}(\tau) = 2$ . Moreover, both  $\sigma$  and  $\tau$  can be assumed to fix an edge and reverse an invariant edge.*

*Proof.* Given  $\varphi : \Gamma \rightarrow \Gamma$  as in the statement, the same recursive construction of the proof of Proposition 3.3 will provide a sequence of automorphisms  $\varphi_i : \Gamma_i \rightarrow \Gamma_i$  and a sequence of subgraphs  $\Lambda_i \subset \Gamma_i$  with  $i = 0, \dots, r$ , satisfying the properties required in that proof, except for the following facts:

- a)  $\Lambda_1$  is allowed to be a subgraph of the first barycentric subdivision of  $\Gamma_1$  instead that a subgraph of  $\Gamma_1$  itself;
- b)  $U_i$  is required to have cardinality  $n$  for  $i = 1, \dots, r-1$  (Lemma 2.2 ensures that such cardinality is either  $n$  or  $n/3$ ).

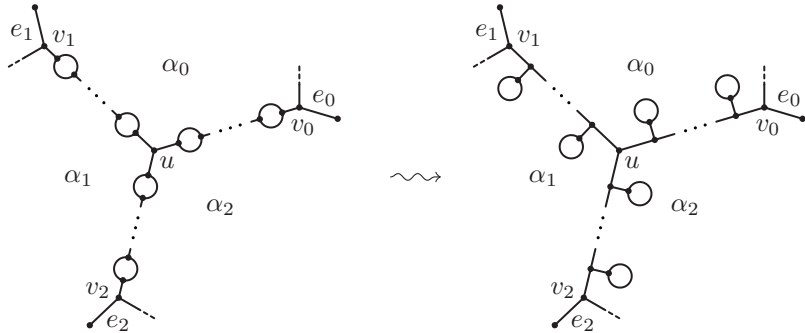


FIGURE 20. The starting configuration  $T$  in the proof of Lemma 4.1.

In particular, all the  $\varphi_i$  are  $F$ -equivalent to  $\varphi$ , and all the restrictions  $\varphi_i|_{\Lambda_i}$  admit a factorization  $\tau_i \circ \sigma_i$  into two involutions of  $\Lambda_i$ . Hence,  $\varphi$  turns out to be  $F$ -equivalent to  $\varphi_r = \tau_r \circ \sigma_r$  (being  $\Lambda_r = \Gamma_r$ ), which proves the proposition.

The starting step of the recursion is provided by Lemma 4.1. If the situation described in point 1 of that lemma occurs, we define the automorphism  $\varphi_1 : \Gamma_1 \rightarrow \Gamma_1$ , the subgraph  $\Lambda_1 \subset \Gamma_1$ , the two involutions  $\sigma_1, \tau_1 : \Lambda_1 \rightarrow \Lambda_1$  and the vertex  $u_1$ , as in the proof of Proposition 3.3. Otherwise, the situation described in point 2 of Lemma 4.1 occurs. In this case, there exists a  $\varphi$ -invariant disjoint union of  $n/3$  tripodes  $T \sqcup \varphi(T) \sqcup \dots \sqcup \varphi^{n/3-1}(T) \subset \Gamma$ , with  $\varphi^{n/3}$  cyclically permuting the edges of the tripode  $T$ . Then, we put  $\Gamma_1 = \Gamma_0 = \Gamma$ ,  $\varphi_1 = \varphi_0 = \varphi$  and  $\Lambda_1 = T' \sqcup \varphi(T') \sqcup \dots \sqcup \varphi^{n/3-1}(T') \subset \Gamma_1$ , where  $T' \subset T$  is the star of the trivalent vertex of  $T$  in the first barycentric subdivision. Moreover, we denote by  $v$  the trivalent vertex of  $T'$  and by  $u$  any free end of  $T'$ , and we define  $\sigma_1$  and  $\tau_1$  by putting  $\sigma_1(\varphi^j(v)) = \varphi^{-j}(v)$ ,  $\sigma_1(\varphi^j(u)) = \varphi^{-j}(u)$ ,  $\tau_1(\varphi^j(v)) = \varphi^{1-j}(v)$  and  $\tau_1(\varphi^j(u)) = \varphi^{1-j}(u)$  for  $j = 0, \dots, n-1$ .

In order to conclude the starting step, it is enough to observe that:  $\sigma_1$  and  $\tau_1$  are well-defined involutive automorphisms of  $\Lambda_1$ , since they preserve the adjacency of vertices, being  $\varphi^j(v)$  adjacent to  $\varphi^k(u)$  if and only if  $j = k \pmod{n/3}$ ;  $\tau_1 \circ \sigma_1$  coincides with the restriction  $\varphi_1|_{\Lambda_1}$ ;  $\Delta_1 = \text{Cl}(\Gamma_1 - \Lambda_1)$  is a uni/trivalent graph;  $U_1 = \Delta_1 \cap \Lambda_1$  consists of the  $\varphi_1$ -orbit of the univalent vertex  $u$  of  $\Delta_1$  and it has cardinality  $n$ ;  $u_1 = u$  is the only vertex of  $U_1$  fixed by  $\sigma_1$ .

Now, to realize the recursive step of the construction, assume we are given  $\varphi_{i-1} : \Gamma_{i-1} \rightarrow \Gamma_{i-1}$ ,  $u_{i-1} \in U_{i-1} \subset \Lambda_{i-1} \subsetneq \Gamma_{i-1}$  and  $\sigma_{i-1}, \tau_{i-1} : \Lambda_{i-1} \rightarrow \Lambda_{i-1}$  with the properties required in the proof of Proposition 3.3 for  $i-1 < r$ , integrated by the points  $a$  and  $b$  said above. In particular, the  $\varphi_{i-1}$ -orbit  $U_{i-1}$  has cardinality  $n$ . Therefore, since  $\text{ord}(\varphi_{i-1}) = n$  by Lemma 2.1, also the restriction of  $\varphi_{i-1}$  to the uni/trivalent graph  $\Delta_{i-1} = \text{Cl}(\Gamma_{i-1} - \Lambda_{i-1})$  is an automorphism of the same order  $n$ . Furthermore, the terminal edges of  $\Delta_{i-1}$  ending at vertices in  $U_{i-1}$  are all distinct (see proof of Proposition 3.3).

If  $\Delta_{i-1}$  does not contain any path joining different terminal edges ending at vertices in  $U_{i-1}$ , then we can terminate the recursion with the  $i$ -th step, in the same way as in the proof of Proposition 3.3.

Otherwise, if a path  $\alpha \subset \Delta_{i-1}$  exists joining different terminal edges of  $\Delta_{i-1}$  ending at vertices in  $U_{i-1}$ , then we can choose it to be minimal (hence simple) and apply Lemma 4.2 to the restriction of  $\varphi_{i-1}$  to  $\Delta_{i-1}$  and to such a minimal  $\alpha$ , in order to get the structure described in that statement for  $\cup_j \varphi_{i-1}^j(\alpha) \subset \Delta_{i-1}$  up to  $F$ -equivalence. Such an  $F$ -equivalence only involves internal edges of  $\Delta_{i-1}$ , hence it does not change the set of free ends  $U_{i-1} \subset \Delta_{i-1}$  and the restriction of  $\varphi_{i-1}|_{\Delta_{i-1}}$ . Therefore, it can be extended to an  $F$ -equivalence of the whole  $\varphi_{i-1}$  on  $\Gamma_{i-1}$ , which leaves  $\Lambda_{i-1}$  and the restriction  $\varphi_{i-1}|_{\Lambda_{i-1}}$  unchanged.

As a result, we get a new uni/trivalent graph  $\Gamma_i$ , such that  $\Lambda_{i-1} \subset \Gamma_i$  and a new automorphism  $\varphi_i : \Gamma_i \rightarrow \Gamma_i$ , which is  $F$ -equivalent to  $\varphi_{i-1}$  and coincides with  $\varphi_{i-1}$  on  $\Lambda_{i-1}$ . We also get a new minimal path  $\alpha \subset \text{Cl}(\Gamma_i - \Lambda_{i-1})$ , which joins different  $\varphi_i$ -equivalent terminal edges of  $\text{Cl}(\Gamma_i - \Lambda_{i-1})$  ending at vertices in  $U_{i-1}$ , such that  $\cup_j \varphi_i^j(\alpha)$  itself (no more up to  $F$ -equivalence) has the structure stated in Lemma 4.2.

Then, we proceed by distinguishing the different situations described in points 1 to 3 of that lemma.

If the situation of point 1 occurs, then the  $i$ -th step is the same as in the proof of Proposition 3.3. In particular, if such step is not the concluding one, then  $U_i$  can be easily seen to have cardinality  $n$ .

In the case when the situation of point 2 occurs, we denote by  $e \subset \Gamma_i$  the terminal edge of  $\text{Cl}(\Gamma_i - \Lambda_{i-1})$  whose free end is  $u_{i-1}$ . Then, we choose  $u_i$  to be the other end of  $e$  and put  $\Lambda_i = \Lambda_{i-1} \cup_j \varphi_i^j(e) \cup_j \varphi_i^j(T)$ , in such a way that  $\Delta_i = \Gamma_i - \Lambda_i$  is a uni/trivalent graph, which meets  $\Lambda_i$  at the  $\varphi_i$ -orbit  $U_i$  of  $u_i$ . Moreover, we define  $\sigma_i$  and  $\tau_i$  to be the unique automorphisms of  $\Lambda_i$  extending  $\sigma_{i-1}$  and  $\tau_{i-1}$  respectively. The existence of such extensions and all the required properties of them, in particular the equality  $\tau_i \circ \sigma_i = \varphi_i|_{\Lambda_i}$  and the fact that  $u_i$  is the unique vertex of  $U_i$  fixed by  $\sigma_i$ , can be proved by the same argument as in the proof of Lemma 3.3.

Finally, if the situation of point 3 occurs, we just put  $\Lambda_i = \Lambda_{i-1} \cup_j \varphi_i^j(\widehat{T}) = \Gamma_i$ , where the last equality is due to the connectedness of  $\Gamma_i$ , and define  $\sigma_i$  and  $\tau_i$  to be the unique extensions of  $\sigma_{i-1}$  and  $\tau_{i-1}$  to  $\Gamma_i$ . These can be easily seen to verify all the required properties for  $i = r$ , hence the recursion terminates with this step.  $\square$

## 5. THE CASE OF ORDER $2^m$

In order to deal with automorphisms  $\varphi : \Gamma \rightarrow \Gamma$  in  $\mathcal{A}_{g,b}$  of order  $2^m$ , we first prove that we can limit ourselves to consider the case when  $\text{ord}_\varphi(e) = 2^m$  for every edge  $e$  of  $\Gamma$  (hence, according to Lemma 2.2,  $\text{ord}_\varphi(v) = 2^m$  for every vertex  $v$  of  $\Gamma$ ).

**Lemma 5.1.** *Let  $\varphi : \Gamma \rightarrow \Gamma$  be an automorphism in  $\mathcal{A}_{g,b}$  with  $\text{ord}(\varphi) = 2^m$ , and let  $2^k = \min_{e \in \Gamma} \text{ord}_\varphi(e)$ , where  $e$  varies among all the edges of  $\Gamma$ . Then, up to  $F$ -equivalence and composition with elementary automorphisms,  $\varphi$  can be reduced to an automorphism  $\psi : \Gamma' \rightarrow \Gamma'$  such that  $\text{ord}(\psi) = 2^k$  and  $\text{ord}_\psi(e) = 2^k$  for every edge  $e$  of  $\Gamma'$ .*

*Proof.* The proof is by induction on the pair  $(m, n_m)$  with respect to the lexicographic order, where  $n_m$  is the number of the edges  $e$  of  $\Gamma$  such that  $\text{ord}_\varphi(e) = 2^m$ .

The base of the induction is for  $m = k$ , and hence  $\text{ord}_\varphi(e) = 2^m$  for all the edges  $e$  of  $\Gamma$ , that is  $n_m = 3g - 3 + 2b$ .

If on the contrary  $m > k$ , then Lemma 2.2 tells us that there are three distinct edges  $e_0, e_1$  and  $e_2$  of  $\Gamma$  sharing a trivalent vertex  $v$  of  $\Gamma$ , such that  $\text{ord}_\varphi(e_0) = \text{ord}_\varphi(v) = 2^{m-1}$ ,  $\text{ord}_\varphi(e_1) = \text{ord}_\varphi(e_2) = 2^m$  and  $e_2 = \varphi^{2^{m-1}}(e_1)$  (note that  $e_1 \neq e_2$ , otherwise  $e_1 = e_2$  would be a loop and its order should be  $2^{m-1}$ ). Let  $v_1$  and  $v_2$  the other ends of  $e_1$  and  $e_2$  respectively.

If  $v_1$  and  $v_2$  are (distinct) univalent vertices, then by composing with the terminal switch  $S_{e_1, e_2}$  we get an automorphism  $\rho = S_{e_1, e_2} \circ \varphi$  with either  $m(\rho) < m(\varphi)$  or  $m(\rho) = m(\varphi)$  and  $n_m(\rho) < n_m(\varphi)$ . In fact,  $\text{ord}_\rho(e) = \text{ord}_\varphi(e)/2$  for all the edges  $e$  in the  $\varphi$ -orbit of  $e_1$  (and  $e_2$ ), while  $\text{ord}_\rho(e) = \text{ord}_\varphi(e)$  for all the other edges of  $\Gamma$ .

If  $v_1$  and  $v_2$  are coinciding trivalent vertices, then the same argument as above applies, but with  $S_{e_1, e_2}$  an internal switch.

Finally, assume that  $v_1$  and  $v_2$  are distinct trivalent vertices. Then, we have the situation depicted in the left side of Figure 21, where  $\varphi^{2^{m-1}}(a_i) = a_{3-i}$  and  $\varphi^{2^{m-1}}(b_i) = b_{3-i}$  for  $i = 1, 2$  (with the four edges  $a_1, a_2, b_1$  and  $b_2$  not necessarily

distinct). In this case, we can perform the  $\varphi$ -invariant move  $F_{\mathcal{T}, \mathcal{T}'}$  acting on a neigh-

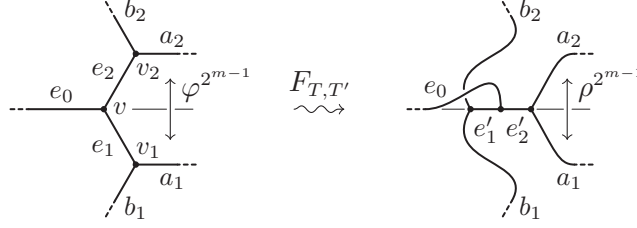


FIGURE 21. Reducing  $(m(\varphi), n_m(\varphi))$ .

borhood of  $e_1 \cup e_2$  as described in Figure 21. This changes  $\varphi$  into an automorphism  $\rho$  with either  $m(\rho) < m(\varphi)$  or  $m(\rho) = m(\varphi)$  and  $n_m(\rho) < n_m(\varphi)$ . In fact  $F_{\mathcal{T}, \mathcal{T}'}$ , replaces the  $\varphi$ -orbit of  $e_1$  (and  $e_2$ ) by two different  $\rho$ -orbits of  $e'_1$  and  $e'_2$  respectively, having cardinality  $2^{m-1}$ . This completes proof of the inductive step.  $\square$

In the light of the previous lemma, the next two lemmas concern the special case of an automorphism  $\varphi : \Gamma \rightarrow \Gamma$  with  $\text{ord}(\varphi) = n = 2^m$  and  $\text{ord}_\varphi(e) = n$  for every edge  $e$  of  $\Gamma$ . We notice that, under the latter assumption, the situation described in Lemma 2.6 can occur only with  $\varphi^i(\alpha) = \varphi^j(\alpha)$ , that is  $i = j \pmod n$ .

**Lemma 5.2.** *Let  $\varphi : \Gamma \rightarrow \Gamma$  be an automorphism of a (possibly disconnected) uni/trivalent graph  $\Gamma$ , with  $\text{ord}(\varphi) = n = 2^m$  and  $\text{ord}_\varphi(e) = n$  for every edge  $e$  of  $\Gamma$ . If  $\alpha \subset \Gamma$  is a (simple) path of minimal length among all paths joining any two distinct  $\varphi$ -equivalent vertices, then  $\cup_i \varphi^i(\alpha)$  is one of the following:*

- 1) a disjoint union  $C \sqcup \varphi(C) \sqcup \dots \sqcup \varphi^{n/\ell-1}(C)$  of  $n/\ell$  simple cycles (possibly a single one, for  $\ell = n$ ), each given by a concatenation of images of  $\alpha$ , having the same structure as described in Lemma 3.1; in this case, up to  $F$ -equivalence we can assume  $\alpha$  to consist of a single edge  $a$  (see Figure 22, where  $\ell$  is a power of 2,  $a_i$  stands for  $\varphi^{is}(a)$  and a similar notation is adopted for the vertices  $v_i$  as well);

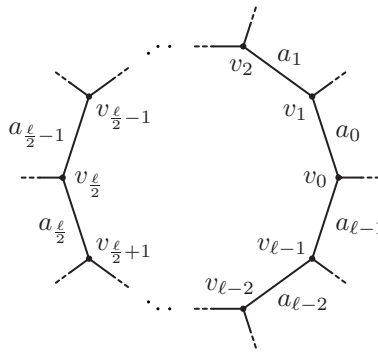


FIGURE 22. The form of the cycle  $C$  in Lemma 5.2.

- 2) a disjoint union  $D \sqcup \varphi(D) \sqcup \dots \sqcup \varphi^{n/2-1}(D)$  of  $n/2$  “diagonal” edges (possibly a single one, for  $n = 2$ ), where  $D = \alpha$  and  $\varphi^{n/2}(D) = \bar{D}$  (see Figure 23, where the vertices  $v_0$  and  $v_1$  are switched by  $\varphi^{n/2}$ , and they can be either both univalent as on the left side or both trivalent as on the right side).

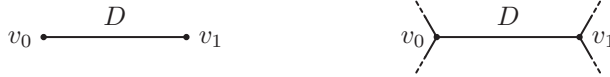


FIGURE 23. The “diagonal” edge  $D$  in Lemma 5.2.

*Proof.* Let  $\alpha \subset \Gamma$  be an arbitrary path of minimal length among all paths joining any two distinct  $\varphi$ -equivalent vertices and let  $v \neq v'$  be its ends. For any  $i \neq j \pmod n$ , let us consider the two paths  $\varphi^i(\alpha)$  and  $\varphi^j(\alpha)$ , and the four different situations described in Lemmas 2.3 to 2.6, which cover all the possibilities, thanks to the  $\varphi$ -equivalence of  $v$  and  $v'$ .

The situation of Lemma 2.6 cannot occur, due to the choice of  $i$  and  $j$ , while any of the three remaining situations may occur. In particular,  $\varphi^i(\alpha)$  and  $\varphi^j(\alpha)$  can have non-empty intersection both as in Lemma 2.5 and as in Lemma 2.4.

Let us assume first that  $\varphi^i(\alpha)$  and  $\varphi^j(\alpha)$  are as in Lemma 2.5, i.e. they share both ends, with  $\varphi^i(v) = \varphi^j(v')$  and  $\varphi^j(v) = \varphi^i(v')$ . In particular,  $\varphi^{j-i}$  switches  $v$  and  $v'$ , and  $j - i = n/2 \pmod n$ , since  $\text{ord}_\varphi(v) = n$ .

The global minimality of  $\alpha$  ensures that either  $\alpha$  is a single “diagonal” edge  $D$  with  $\varphi^{n/2}(D) = \bar{D}$ , or  $\alpha$  and  $\varphi^{n/2}(\alpha)$  only meet at their common ends  $v$  and  $v'$ .

Moreover,  $\alpha$  and  $\varphi^{n/2}(\alpha)$  are disjoint from any other  $\varphi^k(\alpha)$  with  $k \neq 0, n/2 \pmod n$ , thanks to Lemma 2.3. If  $\alpha$  is a single “diagonal” edge  $D$ , we have that  $\cup_i \varphi^i(\alpha)$  is as in point 2 of the statement. Otherwise,  $\cup_i \varphi^i(\alpha)$  consists in the disjoint union  $C \sqcup \varphi(C) \sqcup \dots \sqcup \varphi^{n/2-1}(C)$  of  $n/2$  simple cycles (possibly a single one, for  $n = 2$ ), with  $C = \alpha \cup \varphi^{n/2}(\alpha)$ . Then, arguing as in the proof of Lemma 3.1, the length of  $\alpha$  (and that of its images as well) may be reduced to 1, which gives a special case of the situation described in point 1 of the statement.

At this point, we are left with the case when  $\varphi^i(\alpha)$  and  $\varphi^j(\alpha)$  can only meet as in Lemma 2.4. In this case, denoting by  $\varphi^i(u)$  with  $u \in \alpha$  the common end of  $\delta_{i,1}$  and  $\delta_{j,k}$ , we have that  $\varphi^j(u) \in \delta_{j,k}$  since  $\text{len}(\delta_{i,1}) = \text{len}(\delta_{j,k}) \geq \text{len}(\alpha)/2$ . Actually,  $\varphi^j(u)$  has to coincide with either  $\varphi^j(v')$  or  $\varphi^i(u)$ , otherwise the global minimality of  $\alpha$  would be contradicted. However,  $\varphi^j(u) = \varphi^i(u)$  is not possible because this would imply that  $\text{ord}_\varphi(u) < n$ . Thus,  $\varphi^j(u) = \varphi^j(v')$  and we can conclude that the set  $\cup_i \varphi^i(\alpha)$  has the structure described in point 1 of the statement, by arguing as in the proof of Lemma 3.1.  $\square$

**Lemma 5.3.** *Let  $\varphi : \Gamma \rightarrow \Gamma$  be an automorphism of a (possibly disconnected) uni/trivalent graph  $\Gamma$ , with  $\text{ord}(\varphi) = n = 2^m$  and  $\text{ord}_\varphi(e) = n$  for every edge  $e$  of  $\Gamma$ . If  $\alpha \subset \Gamma$  is a (simple) path of minimal length among all paths joining any two distinct terminal edges in a given  $\varphi$ -orbit, then up to  $F$ -equivalence we can assume  $\cup_i \varphi^i(\alpha)$  to be one of the following:*

- 1) *of a disjoint union  $C \sqcup \varphi(C) \sqcup \dots \sqcup \varphi^{n/\ell-1}(C)$  of  $n/\ell$  simple cycles (possibly a single one for  $\ell = n$ ), each given by a concatenation of images of  $\alpha$ , having the same structure as described in Lemma 3.2, with  $\alpha$  consisting of either one edge  $a$  or two edges  $a$  and  $b$  (see Figure 24, where  $\ell$  is a power of 2,  $a_i$  and  $b_i$  stand for  $\varphi^{is}(a)$  and  $\varphi^{is}(b)$  respectively, and a similar notation is adopted for the terminal edges  $e_i$ 's and the vertices  $v_i$ 's as well);*

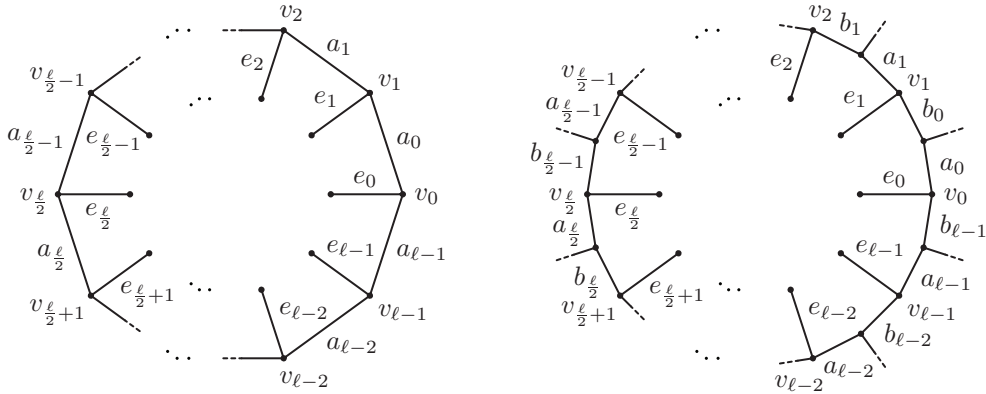


FIGURE 24. The form of the cycle  $C$  in Lemma 5.3.

- 2) *a disjoint union  $D \sqcup \varphi(D) \sqcup \dots \sqcup \varphi^{n/2-1}(D)$  of  $n/2$  “diagonal” edges (possibly a single one, for  $n = 2$ ), where  $D = \alpha$  and  $\varphi^{n/2}(D) = \bar{D}$  (see Figure 25, where the edges  $e_0$  and  $e_1$  are switched by  $\varphi^{n/2}$ ).*

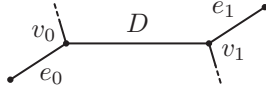


FIGURE 25. The “diagonal” edge  $D$  in Lemma 5.3.

*Proof.* Let  $\alpha$  be a path as in the statement, and let  $e \neq e'$  be the terminal edges it joins. Then the ends of  $\alpha$  coincide with the unique trivalent ends  $v$  and  $v'$  of  $e$  and  $e'$  respectively. Notice that  $v \neq v'$  since otherwise  $\text{ord}_\varphi(v)$  would be less than  $n$ .

For  $i \neq j \pmod n$ , the two paths  $\varphi^i(\alpha)$  and  $\varphi^j(\alpha)$  can meet both as in Lemma 2.4 and as in Lemma 2.5, but not as in Lemma 2.6.

Let us assume first that  $\varphi^i(\alpha)$  and  $\varphi^j(\alpha)$  are as in Lemma 2.5, i.e. they share both ends, with  $\varphi^i(v) = \varphi^j(v')$  and  $\varphi^j(v) = \varphi^i(v')$ , and their union  $\varphi^i(\alpha) \cup \varphi^j(\alpha)$  looks like in Figure 9. In particular,  $\varphi^{j-i}$  switches  $v$  and  $v'$ , and then  $j - i = n/2 \pmod n$ , since  $\text{ord}_\varphi(v) = n$ . Moreover, Lemma 2.3 ensures that  $\varphi^i(\alpha) \cup \varphi^j(\alpha)$  is disjoint from any other path  $\varphi^k(\alpha)$  with  $k \neq i, j \pmod n$ .

Hence,  $\cup_i \varphi^i(\alpha)$  is a disjoint union of  $n/2$  pairs of “diagonal” paths, each pair being given by  $\varphi^i(\alpha)$  and  $\varphi^{i+n/2}(\alpha)$  for some  $i = 0, \dots, n/2 - 1$ . Figure 26 shows the two possible forms of  $\varphi^i(\alpha) \cup \varphi^{i+n/2}(\alpha)$ , depending on the fact that  $\alpha$  and  $\varphi^{n/2}(\alpha)$  start and end with non-trivial common subpaths (left side) or not (right side). Here, the subscripts denote the images under the corresponding power of  $\varphi$ , and apart from the edges  $e_i$  and  $e_{i+n/2}$  each arc represents a path of edges. We notice that in both cases, if the length of the path  $\alpha$  is even, then the central vertices of the paths  $\alpha_i$  and  $\alpha_{i+n/2}$  have to be distinct (that is, they have to belong to non-common subarcs), otherwise the order of such vertices would be  $n/2$ .

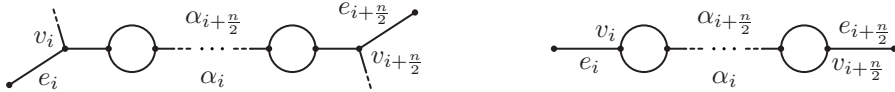


FIGURE 26. The starting form of  $\varphi^i(\alpha) \cup \varphi^{i+n/2}(\alpha)$  in the proof of Lemma 5.3.

The configuration on the left side of Figure 26 can be simplified by a sequence of symmetric pairs of  $\varphi$ -invariant slidings as indicated in Figure 27 (cf. Figure 15), in order to reduce all the arcs to single edges. After that, all the resulting bigons, except the central one if they are odd in number, can be modified in pairs as in Figure 20 and then slid out of  $\varphi^i(\alpha) \cup \varphi^{i+n/2}(\alpha)$  as above. Then, in the case of an even number of bigons we end up with the situation described in point 1 of the statement. Otherwise, we are left with one bigon as shown on the left side of Figure 28, and we can perform the  $F$ -move described in that figure, to get the

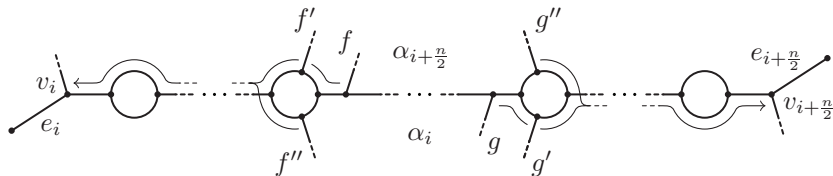


FIGURE 27. Simplifying the configuration on the left side of Figure 26.

configuration on the right side of the same Figure 28. This is the special case for  $\ell = 2$  of the configuration on the right side of Figure 24 considered in point 1 of the statement.

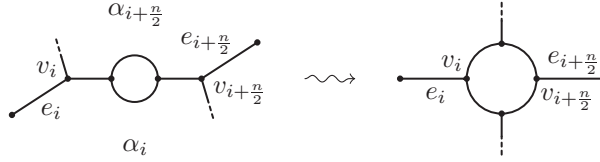


FIGURE 28. Simplified version of the configuration on the left side of Figure 26.

Now, look at the configuration on the right side of Figure 26. If  $\text{len}(\alpha) = 1$ , that is  $\varphi^i(\alpha) \cup \varphi^{i+n/2}(\alpha)$  consists of a single bigon, then we just have the special case for  $\ell = 2$  of the configuration on the left side of Figure 24 considered in point 1 of the statement. If  $\text{len}(\alpha) > 1$ , then we can simplify the configuration by the same argument as above, but sliding everything to the most external loops adjacent to  $e_i$  and  $e_{i+n/2}$ . The result is shown on the left side of Figure 29. We can change it into the configuration on the right side of the figure, by a  $\varphi$ -invariant  $F$ -move on the central edge, and then to the configuration on the right side of Figure 28, by a further  $\varphi$ -invariant sliding (reducing the length of  $\alpha$  to 2). So, we get once again a special case of the situation described in point 1 of the statement.

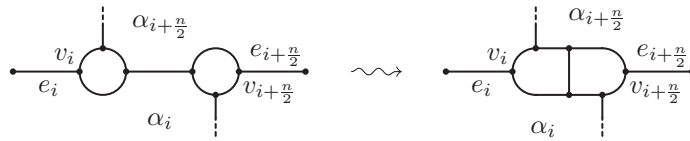


FIGURE 29. Simplified version of the configuration on the right side of Figure 26.

At this point, we are left with the case when  $\varphi^i(\alpha)$  and  $\varphi^j(\alpha)$  can only meet as in Lemma 2.4. In this case, then the fact that  $n$  is even ensures that their common end  $\varphi^j(v) = \varphi^i(v')$  cannot be shared by any other  $\varphi^h(\alpha)$  and that  $\text{len}(\delta_{i,1}) = \text{len}(\delta_{j,k}) > \text{len}(\alpha)/2$  (cf. Figure 7). Therefore, the same arguments as in Lemma 3.1 ensure that the situation can be reduced to the one described in point 1 of the statement.  $\square$

**Proposition 5.4.** *Let  $\varphi : \Gamma \rightarrow \Gamma$  be a non-trivial automorphism in  $\mathcal{A}_{g,b}$ , with  $\text{ord}(\varphi) = n = 2^m$ . Then, up to  $F$ -equivalence and composition with elementary automorphisms, we can reduce  $\varphi$  to the composition  $\tau \circ \sigma$  of two automorphisms  $\sigma, \tau \in \mathcal{A}_{g,b}$  such that  $\text{ord}(\sigma) = \text{ord}(\tau) = 2$ . Moreover,  $\sigma$  can be assumed to fix an edge and  $\tau$  can be assumed to reverse an invariant edge.*

*Proof.* Given  $\varphi : \Gamma \rightarrow \Gamma$  as in the statement, thanks to Lemma 5.1 we can assume that  $\text{ord}_\varphi(e) = n$  for every edge  $e$  of  $\Gamma$ . Then, the same recursive construction of the proof of Proposition 3.3 will provide a sequence of automorphisms  $\varphi_i : \Gamma_i \rightarrow \Gamma_i$  and a sequence of subgraphs  $\Lambda_i \subset \Gamma_i$  with  $i = 0, \dots, r$ , satisfying the properties required in that proof, except for the following facts:

- a)  $\Lambda_1$  is allowed to be a subgraph of the second barycentric subdivision of  $\Gamma_1$  instead that a subgraph of  $\Gamma_1$  itself;

- b)  $\text{ord}_{\varphi_i}(e) = n$  for every edge  $e$  of  $\Gamma_i$  and every  $i = 1, \dots, r$ ;
- c)  $\sigma_i$  fixes exactly two vertices  $u_i$  and  $\varphi_i^{n/2}(u_i)$  in  $U_i$ , for every  $i = 0, \dots, r-1$ ;
- d)  $\sigma_i$  fixes two edges of  $\Gamma_i$ , but it is not required to reverse any invariant edge, for every  $2 \leq i \leq r$ ;
- e) the  $\tau_i$ 's are not required to fix or reverse any edge, except for  $\tau_r$  (the last one) that has to reverse an invariant edge.

In particular, all the  $\varphi_i$ 's are  $F$ -equivalent to  $\varphi$ , and all the restrictions  $\varphi_i|_{\Lambda_i}$  admit a factorization  $\tau_i \circ \sigma_i$  into two involutions of  $\Lambda_i$ . Hence,  $\varphi$  turns out to be  $F$ -equivalent to  $\varphi_r = \tau_r \circ \sigma_r$  (being  $\Lambda_r = \Gamma_r$ ), which proves the proposition.

The starting step of the recursion is provided by Lemma 5.2. If the situation described in point 1 of that lemma occurs, we define the automorphism  $\varphi_1 : \Gamma_1 \rightarrow \Gamma_1$ , the subgraph  $\Lambda_1 \subset \Gamma_1$ , the two involutions  $\sigma_1, \tau_1 : \Lambda_1 \rightarrow \Lambda_1$  and the vertex  $u_1$ , as in the proof of Proposition 3.3. Otherwise, the situation described in point 2 of Lemma 5.2 occurs. In this case, there exists a  $\varphi$ -invariant disjoint union of  $n/2$  “diagonal” edges  $D \sqcup \varphi(D) \sqcup \dots \sqcup \varphi^{n/2-1}(D) \subset \Gamma$ , with  $\varphi^{n/2}(D) = \bar{D}$ . Then, we put  $\Gamma_1 = \Gamma_0 = \Gamma$ ,  $\varphi_1 = \varphi_0 = \varphi$  and  $\Lambda_1 = D' \sqcup \varphi(D') \sqcup \dots \sqcup \varphi^{n/2-1}(D') \subset \Gamma_1$ , where  $D' \subset D$  is the start of the barycenter of  $D$  in the second barycentric subdivision of  $\Gamma$ . Moreover, we denote by  $u$  any one of the two ends of  $D'$ , and we define  $\sigma_1$  and  $\tau_1$  by putting  $\sigma_1(\varphi^j(u)) = \varphi^{-j}(u)$  and  $\tau_1(\varphi^j(u)) = \varphi^{1-j}(u)$  for  $j = 0, \dots, n-1$ .

In order to conclude the starting step, it is enough to observe that:  $\text{ord}_{\varphi_1}(e) = n$  for every edge  $e$  of  $\Gamma_1$ ;  $\sigma_1$  and  $\tau_1$  are well-defined involutive automorphisms of  $\Lambda_1$ , since they preserve the adjacency of vertices, being  $\varphi^j(u)$  adjacent to  $\varphi^k(u)$  if and only if  $j = k \pmod{n/2}$ ;  $\tau_1 \circ \sigma_1$  coincides with the restriction  $\varphi_1|_{\Lambda_1}$ ;  $\Delta_1 = \text{Cl}(\Gamma_1 - \Lambda_1)$  is a uni/trivalent graph;  $U_1 = \Delta_1 \cap \Lambda_1$  consists of the  $\varphi_1$ -orbit of the univalent vertex  $u$  of  $\Delta_1$  and it has cardinality  $n$ ;  $u_1 = u$  and  $\varphi^{n/2}(u_1)$  are the only vertices of  $U_1$  fixed by  $\sigma_1$ .

Now, to realize the recursive step of the construction, assume we are given  $\varphi_{i-1} : \Gamma_{i-1} \rightarrow \Gamma_{i-1}$ ,  $u_{i-1} \in U_{i-1} \subset \Lambda_{i-1} \subsetneq \Gamma_{i-1}$  and  $\sigma_{i-1}, \tau_{i-1} : \Lambda_{i-1} \rightarrow \Lambda_{i-1}$  with the properties required in the proof of Proposition 3.3 for  $i-1 < r$ , integrated by the points a to e said above. In particular, the restriction of  $\varphi_{i-1}$  to the uni/trivalent graph  $\Delta_{i-1} = \text{Cl}(\Gamma_{i-1} - \Lambda_{i-1})$  is an automorphism of order  $n$ , and the order of each edge of  $\Delta_{i-1}$  with respect to it is  $n$  as well.

If two of the terminal edges of  $\Delta_{i-1}$  ending at vertices in  $U_{i-1}$  coincide, then there are  $n/2$  such terminal edges and each of them is a “diagonal” edge reversed by  $\varphi_{i-1}^{n/2}$ . In this case, denoting by  $D$  the “diagonal” edge between  $u_{i-1}$  and  $\varphi_{i-1}^{n/2}(u_{i-1})$ , we put  $\Gamma_i = \Gamma_{i-1}$ ,  $\varphi_i = \varphi_{i-1}$  and  $\Lambda_i = \Lambda_{i-1} \cup_j \varphi_{i-1}^j(D)$ , where the last equality is due to the connectedness of  $\Gamma_i$ . Moreover, we define  $\sigma_i$  and  $\tau_i$  to be the unique extensions of  $\sigma_{i-1}$  and  $\tau_{i-1}$  to  $\Gamma_i$ . These can be easily seen to exist and to verify all the required properties for  $i = r$ , except for the fact that  $\tau_r$  reverses some invariant edge, which will be proved at the end of the proof. Hence, this step terminates the recursion.

Then, let the terminal edges of  $\Delta_{i-1}$  ending at vertices in  $U_{i-1}$  be all distinct.

If  $\Delta_{i-1}$  does not contain any path joining two different such terminal edges, then we can terminate the recursion with the  $i$ -th step, in the same way as in the proof of Proposition 3.3. Once again, we postpone at the end of the proof the verification that  $\tau_r$  reverses some invariant edge.

Otherwise, if a path  $\alpha \subset \Delta_{i-1}$  exists joining different terminal edges of  $\Delta_{i-1}$  ending at vertices in  $U_{i-1}$ , then we can choose it to be minimal (hence simple) and apply Lemma 5.3 to the restriction of  $\varphi_{i-1}$  to  $\Delta_{i-1}$  and to such a minimal  $\alpha$ , in order to get the structure described in that statement for  $\cup_j \varphi_{i-1}^j(\alpha) \subset \Delta_{i-1}$  up to  $F$ -equivalence. Such an  $F$ -equivalence only involves internal edges of  $\Delta_{i-1}$ , hence it does not change the set of free ends  $U_{i-1} \subset \Delta_{i-1}$  and the restriction of  $\varphi_{i-1}|_{\Delta_{i-1}}$ . Therefore, it can be extended to an  $F$ -equivalence of the whole  $\varphi_{i-1}$  on  $\Gamma_{i-1}$ , which leaves  $\Lambda_{i-1}$  and the restriction  $\varphi_{i-1}|_{\Lambda_{i-1}}$  unchanged.

As a result, we get a new uni/trivalent graph  $\Gamma_i$ , such that  $\Lambda_{i-1} \subset \Gamma_i$  and a new automorphism  $\varphi_i : \Gamma_i \rightarrow \Gamma_i$ , which is  $F$ -equivalent to  $\varphi_{i-1}$  and coincides with  $\varphi_{i-1}$  on  $\Lambda_{i-1}$ . We also get a new minimal path  $\alpha \subset \text{Cl}(\Gamma_i - \Lambda_{i-1})$ , which joins different  $\varphi_i$ -equivalent terminal edges of  $\text{Cl}(\Gamma_i - \Lambda_{i-1})$  ending at vertices in  $U_{i-1}$ , such that  $\cup_j \varphi_i^j(\alpha)$  itself (no more up to  $F$ -equivalence) has the structure stated in Lemma 5.3. Then, we proceed by distinguishing the different situations described in points 1 and 2 of that lemma.

If the situation of point 1 occurs, then the  $i$ -th step is the same as in the proof of Proposition 3.3, but now  $\sigma_i$  fixes two edges and  $\tau_i$  reverses two invariant edges.

In the case when the situation of point 2 occurs, we denote by  $e$  the terminal edge of  $\text{Cl}(\Gamma_i - \Lambda_{i-1})$  whose free end is  $u_{i-1}$ . Then, we choose  $u_i$  to be the other end of  $e$  and put  $\Lambda_i = \Lambda_{i-1} \cup_j \varphi_i^j(e) \cup_j \varphi_i^j(D)$ , in such a way that  $\Delta_i = \text{Cl}(\Gamma_i - \Lambda_i)$  is a uni/trivalent graph, which meets  $\Lambda_i$  at the  $\varphi_i$ -orbit  $U_i$  of  $u_i$ . Moreover, we define  $\sigma_i$  and  $\tau_i$  to be the unique automorphisms of  $\Lambda_i$  extending  $\sigma_{i-1}$  and  $\tau_{i-1}$  respectively. To see that such extensions exist, we first define  $\sigma_i(\varphi_i^j)(u_i) = \varphi_i^{-j}(u_i)$  and  $\tau_i(\varphi_i^j)(u_i) = \varphi_i^{1-j}(u_i)$ , for every  $j = 0, \dots, n-1$ , and then we verify that these definitions, together with the previous definitions of  $\sigma_{i-1}$  and  $\tau_{i-1}$ , preserve adjacency of vertices in  $\Lambda_i$ . Notice that  $\tau_i \circ \sigma_i$  trivially coincides with  $\varphi_i|_{\Lambda_i}$ . Furthermore,  $\sigma_i$  fixes only the vertices  $u_i$  and  $\varphi_i^{n/2}(u_i)$  in the  $\varphi_i$ -orbit  $U_i$ , and it fixes the edges  $e$  and  $\varphi^{n/2}(e)$ . On the contrary,  $\tau_i$  neither fixes nor reverses any edge in the  $\varphi_i$ -orbits of  $e$  and  $D$ . This concludes the recursive step.

Finally, we observe that the situation of point 1 of Lemma 5.3 occurs in at least one of the recursive steps due to the connectedness of  $\Gamma$ , and from that step on all the  $\tau_i$  reverse some invariant edge. In particular, this is true for  $\tau_r$ .  $\square$

At this point, taking into account Propositions 3.3, 4.3 and 5.4, we are left to show that any automorphism in  $\mathcal{A}_{g,b}$  belongs to  $\mathcal{E}_{g,b}$ , in the special case when it has order 2 and fixes or reverses some edge. This is the content of our last proposition.

**Proposition 5.5.** *Let  $\varphi : \Gamma \rightarrow \Gamma$  be an automorphism in  $\mathcal{A}_{g,b}$  of order 2, which either fixes an edge or reverses an invariant edge. Then, up to  $F$ -equivalence and composition with elementary switches, we can reduce  $\varphi$  to the identity.*

*Proof.* If  $\varphi$  fixes an edge  $e$ , then Lemma 5.1 ensures that up to  $F$ -equivalence and composition with elementary automorphisms,  $\varphi$  can be reduced to an automorphism  $\psi : \Gamma' \rightarrow \Gamma'$  such that  $\text{ord}(\psi) = 1$ , that is  $\psi$  reduces to the identity. On the other hand, if  $\varphi$  reverses an invariant edge  $e$ , we can perform a  $\varphi$ -invariant  $F$ -move  $F_{e,e'}$ , in order to get a new automorphism  $\varphi' : \Gamma' \rightarrow \Gamma'$ , which fixes the edge  $e'$  of  $\Gamma'$ .  $\square$

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