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MANIFOLDS AS BRANCHED COVERS OF SPHERES

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The concept of branched covering originated from the theory of ramified surfaces, introduced by Riemann for describing multivalued complex functions.

The basic example arises from the map $z \mapsto w = z^k$ with $k > 1$. This map is singular only at $z = 0$, and its restriction to $z, w \neq 0$ is a cyclic ordinary covering of degree k. In fact the z-plane can be decomposed into k angles each one of which is bijectively mapped onto the w-plane. So, we can think of the z-plane as the union of k sheets over the w-plane, in such a way that the multivalued map $w \mapsto z = \sqrt[k]{w}$ takes a single value on each sheet.

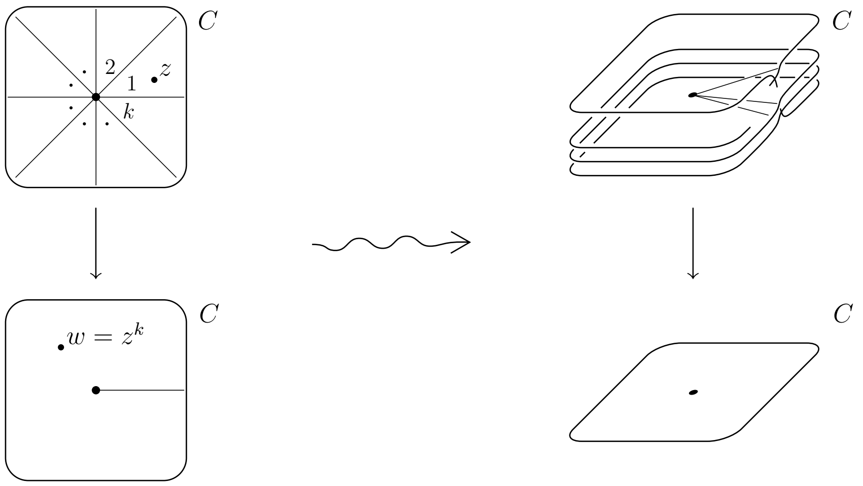

Figure 1.

We denote by $p_k : B^2 \to B^2$ the restriction to the unit disk $B^2 \subset C$ of the complex map $z \mapsto w = z^k$ with $k \geq 1$. The maps p_k will be our local models for branched coverings between surfaces. If $k > 1$ we will call the point $z = 0$ a *singular point* and the point $w = 0$ a *branch point*.

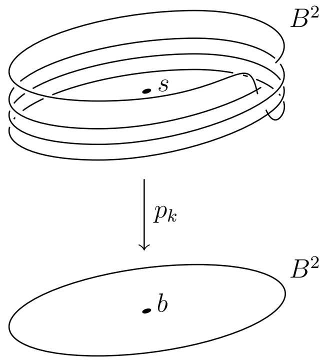

Figure 2.

Before giving the definition of branched covering between surfaces, look at another example. Let $Q \subset CP^2$ a non-singular cubic in the complex projective plane and $\pi : Q \to L$ be the projection of Q onto any complex line $L \subset CP^2$ from a generic point $c \in CP^2$. Then, π is singular at $x \in Q$ if and only if the projecting line cx is tangent to Q at x. So, there are six singular points in Q that are projected by π into six branch points in L, and at each singular point π looks like the double covering p_2. Over each branch point of π there are two points of Q, only one of them is singular the other one is called a *pseudo-singular point*. On the other side, over the complement of the branch points π is an irregular ordinary covering of degree 3.

Now, let A be an arc in L joining two branch points b_1 and b_2, such that $\widetilde{A} = \pi^{-1}(A)$ is an arc in Q joining the corresponding pseudo-singular points s_1' and s_2', as in figure 3. If N is a regular neighborhood of A in L, then $\widetilde{N} = \pi^{-1}(N)$ is a regular neighborhood of $\widetilde{A}$ in Q, and both N and $\widetilde{N}$ are homeomorphic to B^2. So, by restricting π, we get (up to homeomorphism) a 3-fold branched covering $p : B^2 \to B^2$ with two singular points and two branch points.

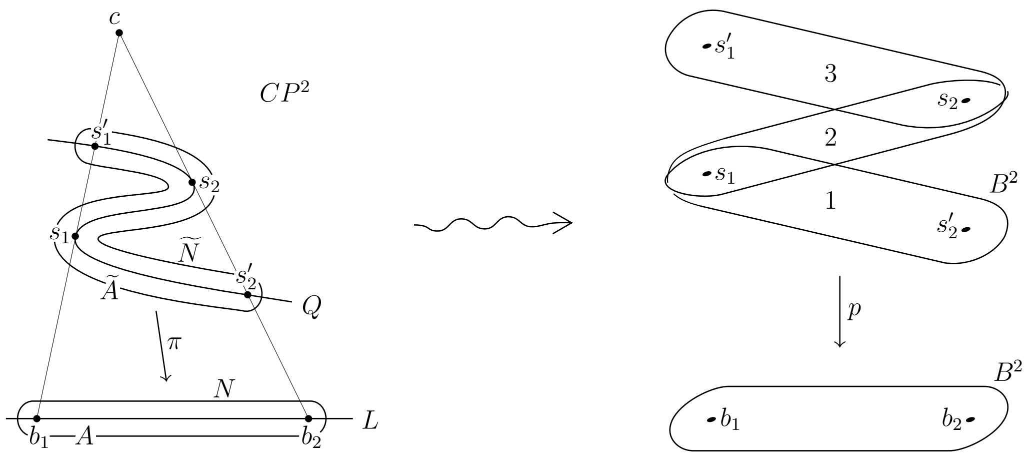

Figure 3.

In the light of these two examples, we give the following:

Definition 1. A map $p : \widetilde{F} \to F$ between closed surfaces is called a *branched covering* if it is finite-to-one and for every $x \in \widetilde{F}$ there exists a neighborhood U of x in $\widetilde{F}$ such that the restriction $f_{|U} : U \to f(U)$ is homeomorphic to $p_{d(p,x)}$ with $d(p,x) \geq 1$. The number $d(p,x)$ is called the *branching index* of p at x. Moreover, we put $S_p = \{x \in \widetilde{F} \mid d(p,x) > 1\}$ (the *singular set* of p), $B_p = p(S_p)$ (the *branch set* of p) and $S_p' = p^{-1}(B_p) - S_p$ (the *pseudosingular set* of p).

By the well-known theory of ordinary coverings, we have the following two important facts, as immediate consequences of the definition:

a) the restriction $p_| : \widetilde{F} - p^{-1}(B_p) \to F - B_p$ is a finite ordinary covering of degree d_p (the *degree* of p), and so it can be described in terms of its monodromy $\omega_p : \pi_1(F - B_p) \to \Sigma_{d_p}$ (the *monodromy* of p);

b) p is completely determined (up to homeomorphism) by such restriction, hence it can be represented by means of the branch set $B_p \subset F$ and the monodromy ω_p.

Then, we can represent the two examples above as shown in figure 4, where each branch point is labelled with the monodromy of a fixed meridian around it:

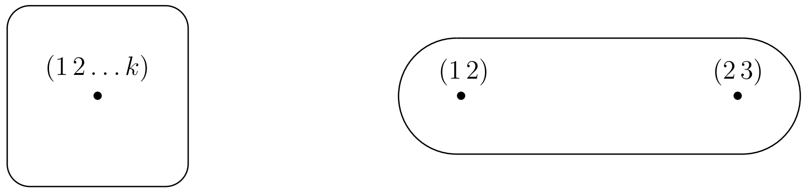

Figure 4.

Now, branched coverings between higher-dimensional manifolds, can be defined analogously, by replacing two-dimensional local models with higher-dimensional ones. We will construct these local models by induction on the dimension.

The most trivial way to do that is crossing local models by the identity of the interval $I = [-1, 1]$. Namely, from a local model $p : (B^{m-1}, 0) \to (B^{m-1}, 0)$ in dimension $m - 1$, we get a local model $p \times \mathrm{id}_I : (B^m, 0) \to (B^m, 0)$ in dimension m, where B^m is identified with $B^{m-1} \times I$.

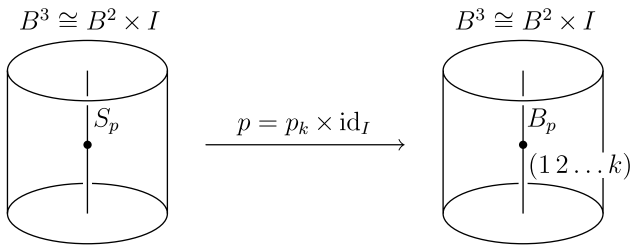

Figure 5.

By iterating the crossing process, starting from the two-dimensional local model p_k, we get a local model $p_k \times \mathrm{id}_{I^{m-2}} : (B^m, 0) \to (B^m, 0)$ in any dimension m. Of course, both the singular set and the branch set of this local model are flat $(m-2)$-cells, and the restriction to the complements of them is a cyclic ordinary covering of degree k (again we call k the local branching index).

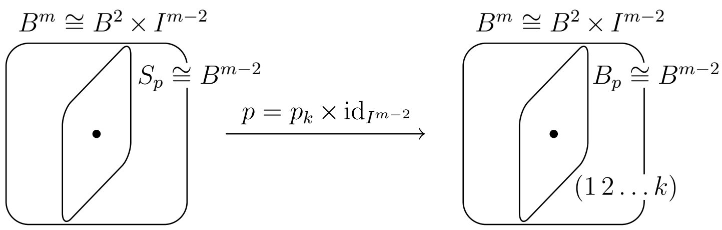

Figure 6.

In the figures 5 and 6, as well as in all the figures that follow, we use labels to indicate the monodromy of meridians around the branch set.

We remark that the singular set of all the branched coverings arising from the local models considered above, is a codimension 2 locally flat submanifold of the covering manifold. Nevertheless, the branch set can be singular, since self-intersections may appear when we map the singular set into the covered manifold. However, we can always assume, by transversality, that the branch set is an immersed manifold with only trasversal self-intersections. In dimension 4, there is only one kind (up to homeomorphism) of such singularities of the branch set, consisting of an isolated trasversal double point, which we call a *node* (see figure 7).

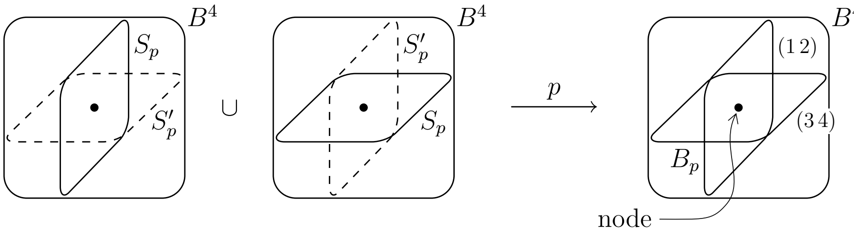

Figure 7.

A more general way to construct a local model in dimension m consists in making the cone of a branched covering between $(m-1)$-spheres. That is, from any branched covering $p : S^{m-1} \to S^{m-1}$ we get a local model $C(p) : B^m \to B^m$, where B^m is identified with the cone $C(S^{m-1})$. Of course, all the local models considered above are cones. Moreover they are the only ones whose singular set is non singular, in fact we have $S_{C(p)} = C(S_p)$ and $B_{C(p)} = C(B_p)$ for any branched covering p between spheres.

In order to construct new local models, we need branched coverings between $(m-1)$-spheres. Such a branched covering can be given by gluing together along S^{m-2} any two branched coverings $p, q : B^{m-1} \to B^{m-1}$ such that: 1) B_p and B_q meet S^{m-2} transversally, 2) $p_{|S^{m-2}} \cong q_{|S^{m-2}}$.

Two examples of local models in dimension 3 are represented (using labelled branch sets) in figure 8: the first one is the cone of the map obtained by gluing together two copies of the covering p depicted in figure 3, the second one is the cone of the map obtained by gluing together one copy of the same covering p with the cyclic covering p_3. In both these examples the origin is a non transversal singularity for the branch set as well as for the singular set.

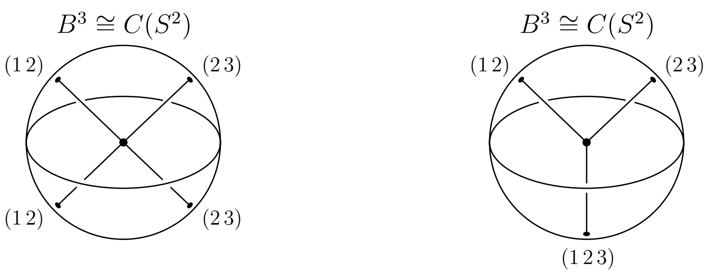

Figure 8.

By gluing together two copies of the covering p of figure 3 crossed by id_I, we get different branched coverings of S^3 by itself, depending on the gluing homeomorphism. Labelled diagrams of the branch sets of two of them are shown in figure 9 (the dashed lines represent the gluing spheres):

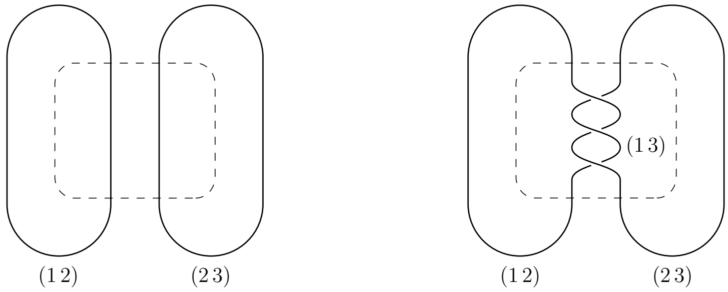

Figure 9.

The cones of these two coverings are local models in dimension 4, whose branch sets respectively have a non-transversal double point and a *cusp* singularity (the cone of a trefoil knot).

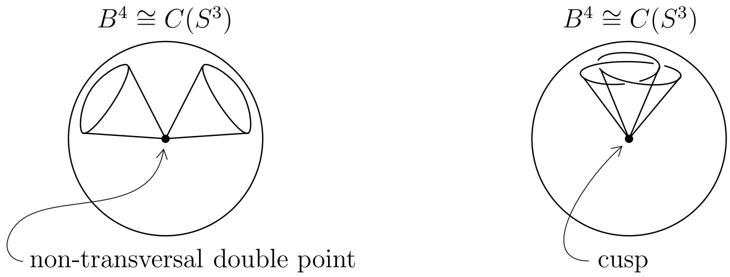

Figure 10.

At this point, we can give the definition of branched covering between higher-dimensional manifolds, by induction on the dimension:

Definition 2. A PL map $p : \widetilde{M} \to M$ between closed PL m-manifolds is called a *branched covering* if it is non-degenerate and for every $x \in \widetilde{M}$ there exists a neighborhood U of x in $\widetilde{M}$ such that the restriction $f_{|U} : U \to f(U)$ is homeomorphic to the cone of a branched covering of S^{m-1} onto itself. Moreover, we put $S_p = \{x \in \widetilde{M} \mid p$ is not locally injective at $x\}$ (the *singular set* of p), $B_p = p(S_p)$ (the *branch set* of p) and $S'_p = \mathrm{Cl}(p^{-1}(B_p) - S_p)$ (the *pseudo-singular set* of p).

Observe that S_p, S'_p and B_p are homogeneously $(m-2)$-dimensional subcomplexes. Furthermore, properties *a*) and *b*) of branched coverings we have seen in the two-dimensional case, still hold in general (cf. [27]):

a) the restriction $p_| : \widetilde{M} - p^{-1}(B_p) \to M - B_p$ is a finite ordinary covering of degree d_p (the *degree* of p), and so it can be described in terms of its monodromy $\omega_p : \pi_1(M - B_p) \to \Sigma_{d_p}$ (the *monodromy* of p);

b) p is completely determined (up to homeomorphism) by such restriction, hence it can be represented by means of the branch set $B_p \subset M$ and the monodromy ω_p.

A more technical definition of branched covering, is usually based on the remark following our definition and on the property *a*).

The first result in the direction of representing manifolds as branched covers of spheres is the following theorem proved by J. M. Alexander [1] in 1920.

Theorem 1. *Any orientable closed PL m-manifold M is a branched cover of S^m.*

Sketch of proof. Let T be a triangulation of M, then the baricentric subdivision βT of T has a black and white chessboard-coloration. Now, a branched covering $p : M \to S^m$ can be easily defined by sending the black m-simplices of βT onto the standard simplex $\Delta^m \subset R^m \subset R^m \cup \{\infty\} \cong S^m$ and the white ones onto $\mathrm{Cl}(S^m - \Delta^m)$. □

Note that the degree of the covering p constructed in the proof of theorem 1 depends on the number of m-simplices of T, moreover the singular set S_p is the $(m-2)$-skeleton of βT and the branch set B_p is the $(m-2)$-skeleton of Δ^m. So, two natural questions naturally arise, in order to make branched covers of spheres an effective tool for representing and studying manifolds:

Question 1. *There exists a bound $d(m)$ for d_p depending only on the dimension m?*

Question 2. *There exists p such that S_p is non-singular?*

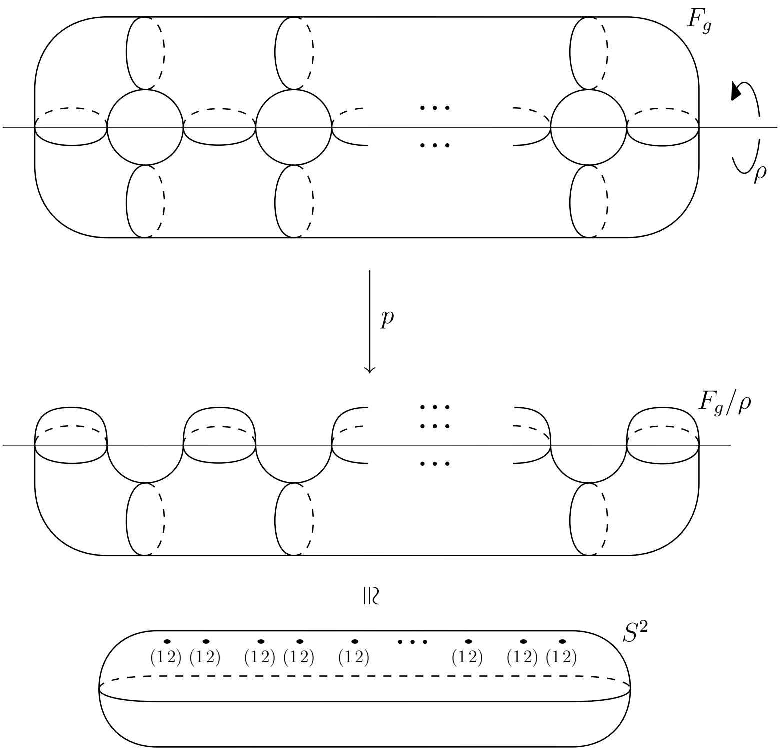

Figure 11.

I. Benstein and A. L. Edmonds in [4] established that if $d(m)$ exists, it cannot be less that m,in fact the m-dimensional torus $T^m \cong S^1 \times \cdots \times S^1$ is a m-fold branched cover of S^m (cf. [47]), but there is no branched covering of T^m onto S^m of degree $< m$. In the same paper, they also prove that in general we cannot require B_p non-singular (the simplest counterexample they give is in dimension 8). We will show that the answer two both the questions is positive for dimensions $m \leq 4$.

The two-dimensional case is trivial: any orientable closed surface F_g of genus g can be represented as a 2-fold covering of S^2 branched over $2g+2$ points, as shown in figure 11.

In dimension 3, the following theorem was proved independently by H. M. Hilden, U. Hirsch and J. M. Montesinos ([36], [49] and [55]):

Theorem 2. *Any orientable closed 3-manifold M is a simple 3-fold covering of S^3 branched over a knot. (simple means that the monodromy of each meridian around the branch knot is a transposition)*

Sketch of proof. First of all, we observe that the branched covering of figure 11 can be extended to a simple branched covering $H_g \to B^3$, where H_g is the handelbody bounded by F_g in R^3. The branch set of this covering consists of $g+1$ trivial arcs in

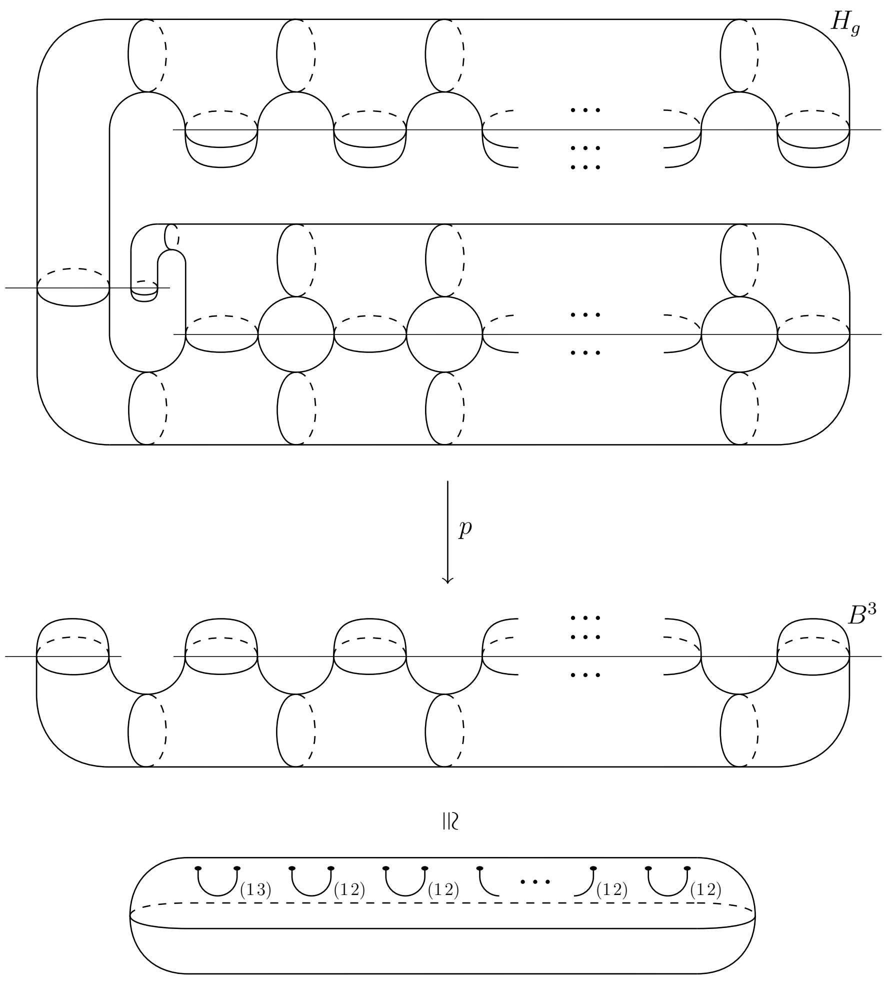

Figure 12.

B^3 (labelled with the transposition $(1\,2)$). By adding an extra trivial arc labelled with $(1\,3)$ to the branch set, we get a new branched covering $H_g \to B^3$ (see figure 13), with the following property: any homeomorphism $k : F_g \to F_g$ is (up to isotopy) the lifting of a homeomorphism $h : S^2 \to S^2$ (cf. [5]). Then, by considering a Heegaard splitting of M, we get a 3-fold simple branched covering $M \cong H_g \cup_k H_g \to B^3 \cup_h B^3 \cong S^3$, as shown in figure 13.

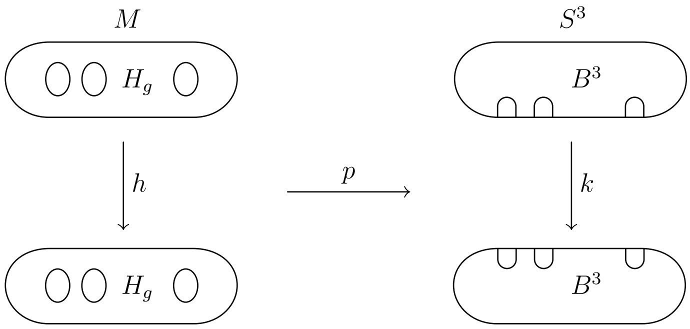

Figure 13.

The branch set of this covering is a link in S^3, which can be represented by a plat, labelled as shown in figure 14. Finally, such a link can be made into a knot, by using moves of type I described in figure 15. □

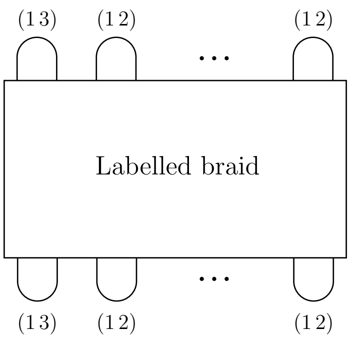

Figure 14.

By this theorem, any orientable closed 3-manifold can be represented as a cover $M(L, \omega)$ of S^3, where we can think of (L, ω) as a *labelled diagram* of a knot (or more generally of a link), that is a diagram whose bridges are labelled by transpositions.

R. Fox and J. M. Montesinos posed the following problem: find a set of moves relating any two labelled diagrams representing the same 3-manifold (up to homeomorphism). For a long time, the move I of figure 15 (together with labelled Reidemeister moves) was erroneously conjectured to be enough (cf. [60]). A complete set of moves was recently given in [71], but the new moves were not completely satisfactory because of their complexity and non-local character.

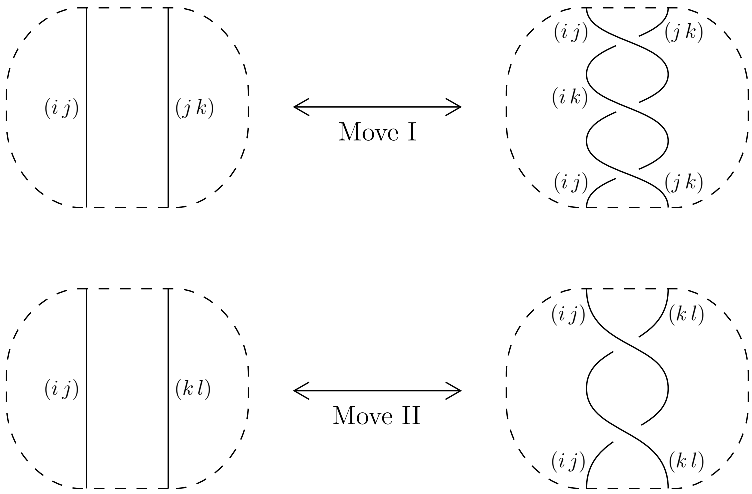

Figure 15.

Finally, in [74] it was proved that such inconvenient can be avoided by stabilizing the coverings as show in figure 16. Namely, we have the following equivalence theorem:

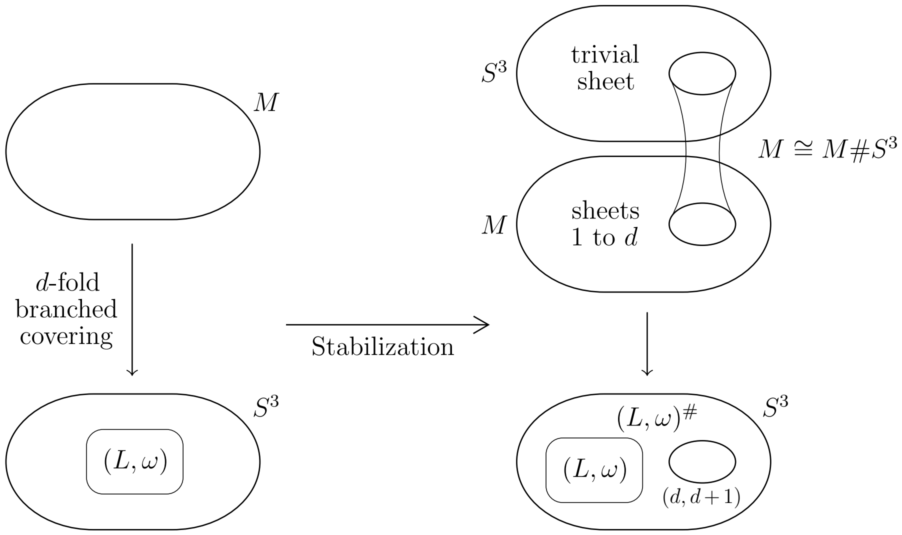

Figure 16.

Theorem 3. *Let* (L,ω) *and* (L',ω') *be labelled link diagrams representing two simple 3-fold branched coverings of* S^3*. Then* $M(L,\omega) \cong M(L',\omega')$ *if and only if the stabilizations* $(L,\omega)^\#$ *and* $(L',\omega')^\#$ *are related by a finite sequence of moves I and II and labelled Reidemeister moves.*

Sketch of proof. First of all, the moves I and II do not change the covering manifold, since the 3-cell that they involve is covered by a disjoint union of 3-cells.

On the other hand, let (L,ω) and (L',ω') be two labelled diagrams representing the same 3-manifold M. We can assume, up to labelled Reidemeister moves, that they are plats as in figure 14, in such a way that they induce two Heegaard splittings of M. Now, move I allows us to realize a stable equivalence between these splittings, in order to get two new labelled plats inducing the same splitting homeomorphism (cf. section 2 of [71]). Finally, the stabilizations of these new labelled plats can be related by moves I and II, since these moves (in presence of the fourth trivial sheet) generate all the braids representing the identity homeomorphism of F_g (cf. section 3 of [71] and [74]). □

The following question remains still open:

Question 3. *Are moves I and II together with stabilization and labelled Reidemeister moves sufficient in order to relate any two labelled link diagrams representing the same* 3*-manifold as a simple* 4*-fold (n-fold) branched cover of* S^3*?*

Finally, in dimension 4 the following representation theorem was proved in [74], by using covering moves.

Theorem 4. *Any orientable closed PL* 4*-manifold is a simple* 4*-fold cover of* S^4 *branched over a transversally immersed surface.*

Sketch of proof. Let M be an orientable closed PL 4-manifold. By [57] and using a handlebody decomposition, we can write $M = M_0 \cup_{\mathrm{Bd}} M_1$ where both M_0 and M_1 are simple 3-fold covers of B^4 branched over locally flat surfaces. Looking at the boundaries, we have two simple 3-fold branched coverings of S^3 by the same 3-manifold $\mathrm{Bd}M_0 = \mathrm{Bd}M_1$. By stabilizing these coverings and relating them by moves, we get a simple 4-fold branched covering $p : M \to S^4$ (see figure 17).

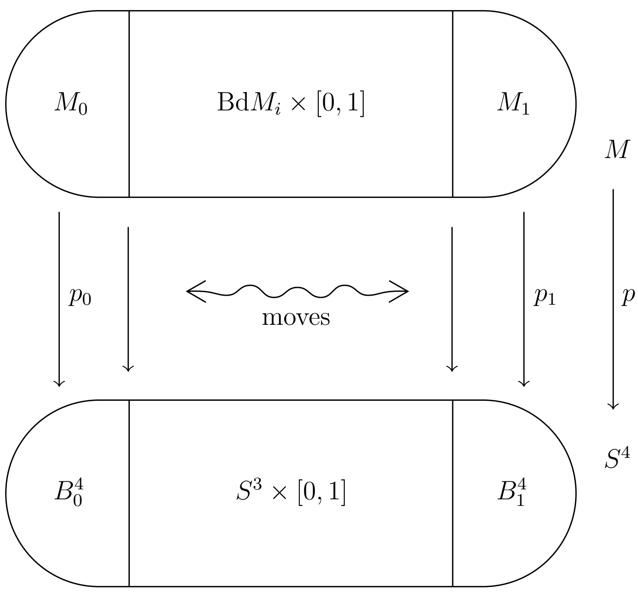

Figure 17.

The branch set of p is a surface $F \subset S^4$, whose only singularities are nodes and cusps coming from the moves I and II as suggested by figure 18. Finally, by using branched covering cobordism (cf. [39]), we can remove all the cusps of F, in order to make it transversally immersed. □

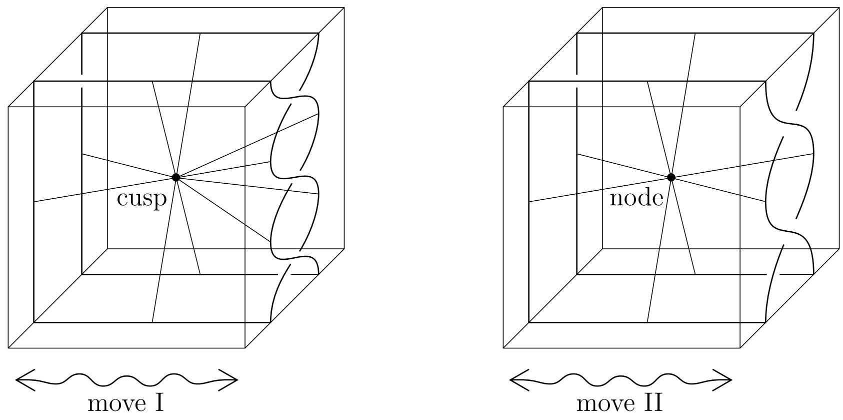

Figure 18.

We conclude by remarking that the singular set of the covering p constructed in the proof of theorem 4 must be a locally flat surface in M, but the following question is still open:

Question 4. *Can any orientable closed PL* 4*-manifold be represented as a simple cover of S^4 branched over a locally flat surface?*

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