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BRANCHFOLDS AND RATIONAL CONIFOLDS

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Abstract

We extend the concept of orbifold to that of branchfold, in order to allow cone singularities with rational angles, and we show why branchfolds naturally fit in the theory of branched coverings. Then, we obtain a geometric goodness theorem for branchfolds and apply it to prove that a conifold can be endowed with a branchfold structure if and only if it has locally finite holonomy.

Keywords: branchfold, orbifold, rational conifold, branched covering.

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Introduction

This paper introduces a class of spaces which provide an algebraic point of view for studying conifolds whose codimension two cone singularities have rational angle of $2k\pi/h$ radians, analogously to what orbifolds do only for angles of $2\pi/h$ radians.

We call these spaces branchfolds, since they naturally fit in the theory of branched coverings. Roughly speaking, an m-branchfold is a space covered by open sets U admitting two regular branched coverings $V \leftarrow P \to U$, with P a polyhedron and V an open subset of R^m, rather than only one $V \to U$ as in the orbifold case.

Since in dimension 3 conifolds with rational angles are dense in the space of all conifolds, at least in the case of cone manifolds with link singularities (cf. [9], [14] and [5]), in principle branchfold could allow an algebraic approach to the deformation theory of conifolds, which is a crucial analytic aspect of the proof of the orbifold geometrization theorem [1]. On the other hand, branchfolds could be useful to shed some light upon the Cheeger-Simons problem of whether the volume of a compact spherical m-dimensional conifold with rational angles is a rational multiple of the volume of the m-sphere (cf. [4] and [6]). Here, we limit ourselves to set up the basic theory of branchfolds and to establish the relation between branchfolds and conifolds, while the above mentioned possible applications will be considered in future papers.

In Section 1 we review the Fox theory of branched covering and prove some preliminary results. The general setting of branchfold spaces and maps is given in Section 2. In particular, Propositions 2.15, 2.16 and 2.17 relate branchfold coverings to branched coverings. Section 3 contains our main results: geometric branchfold are introduced and a geometric goodness theorem (Theorem 3.12) is proved. As a consequence of such a theorem, we can characterize the conifolds which admit a branchfold structure (Theorem 3.15) as those which have locally finite holonomy.

1. Preliminaries

In this section we review some standard theory of branched coverings between polyhedra in the sense of Fox [7], reformulating definitions and basic facts in modern language. Then, we consider the notion of good action and some elementary properties of pseudo-manifolds needed to define branchfolds in the next section.

By a *polyhedron* we mean a topological space P endowed with a *polyhedral structure*, i.e. a PL equivalence class of locally finite countable triangulations. A map $f : P \to Q$ between polyhedra is a *PL map* (resp. a *locally PL map*) if there are triangulations K of P and L of Q such that f sends each simplex of K linearly *onto* (resp. *into*) a simplex of L. In both cases f is called *non-degenerate* if is preserves the dimensions of the simplexes of K. This terminology *is not standard for non-compact polyhedra*, for which PL usually means locally PL in our terms. In particular, according our definitions, the class of PL maps is not closed under composition, while such is the class of locally PL maps. However, it can be shown that locally PL coincides with PL for proper maps. A subspace $S \subset P$ of a polyhedron is a *subpolyhedron* of P if there exists a triangulation K of P and a subcomplex $L \subset K$ such that $S = |L|$. On the other hand, any open subspace of a polyhedron P is understood to be a polyhedron, with the unique polyhedral structure making the inclusion a locally PL map. Given a polyhedron P and a point $x \in P$, we denote by $\mathrm{St}_x P$ (resp. $\mathrm{Lk}_x P$) any *star* (resp. *link*) of x in P, that is the underlying space of the simplicial star $\mathrm{St}(x, K)$ (resp. link $\mathrm{Lk}(x, K)$) of x in any triangulation K of P having x as a vertex. Notice that $\mathrm{St}_x P$ and $\mathrm{Lk}_x P$ are subpolyhedra of P well defined up to pseudo-radial PL homeomorphisms centered at x. Polyhedra have local conical structure. In fact, the stars $\mathrm{St}_x P$ give a basis of *conical neighborhoods* of x in P, being $\mathrm{St}_x P$ the cone of $\mathrm{Lk}_x P$ with apex x. A similar local conical structure is exhibited by non-degenerate (locally) PL maps. Namely, for any non-degenerate (locally) PL map $f : P \to Q$ and any $x \in P$, putting $y = f(x)$ we have that $\mathrm{St}_x f = f_| : \mathrm{St}_x P \to \mathrm{St}_y Q$ is the cone of $\mathrm{Lk}_x f = f_| : \mathrm{Lk}_x P \to \mathrm{Lk}_y Q$. Such restrictions are well defined up to pseudo-radial PL homeomorphisms. By the *local model* of f at x we mean the open cone restriction $\mathrm{St}_x f - \mathrm{Lk}_x f$.

Good subpolyhedra

Let P be a polyhedron. A subpolyhedron $S \subset P$ is called *good* if it is nowhere dense in P and its complement $P - S$ is locally connected at S, meaning that every point $x \in S$ has arbitrarily small neighborhoods N such that $N - S$ is connected.

The elementary properties stated below will be useful in dealing with branched coverings. The first three proofs are almost straightforward, hence we omit them.

LEMMA 1.1. *Let P be a polyhedron and $S \subset P$ be a nowhere dense subpolyhedron. Then S is a good subpolyhedron of P if and only if one of the following equivalent properties holds:*

(1) $\mathrm{St}_x P - S$ is connected $\forall x \in S$; (2) $\mathrm{Lk}_x P - S$ is connected $\forall x \in S$.

PROPOSITION 1.2. *Let P be a polyhedron and $S \subset P$ be a good subpolyhedron. Then any subpolyhedron $R \subset S$ is a good subpolyhedron of P.*

PROPOSITION 1.3. *If P is a connected polyhedron and $S \subset P$ is a good subpolyhedron, then $P - S$ is connected.*

Proposition 1.4. *Let P be a polyhedron and $S \subset P$ be a nowhere dense subpolyhedron. Then S is a good subpolyhedron of P if and only if one of the following equivalent properties holds:*
(1) $\mathrm{St}_x S$ *is a good subpolyhedron of* $\mathrm{St}_x P$ *for every* $x \in S$;
(2) $\mathrm{Lk}_x P$ *is connected and* $\mathrm{Lk}_x S$ *is a good subpolyhedron of* $\mathrm{Lk}_x P$ *for every* $x \in S$.

Proof. Since S is nowhere dense in P, $\mathrm{St}_x S$ and $\mathrm{Lk}_x S$ are nowhere dense respectively in $\mathrm{St}_x P$ and $\mathrm{Lk}_x P$, $\forall x \in S$. We prove that (1) holds when S is good in P. The definition of good subpolyhedron is of local nature. Hence the goodness of $S \subset P$ implies that of the open star $\mathrm{St}_x S - \mathrm{Lk}_x S \subset \mathrm{St}_x P - \mathrm{Lk}_x P$, for every $x \in S$. Then, the goodness of $\mathrm{St}_x S \subset \mathrm{St}_x P$ follows from the conical structure of stars. To prove that (1) implies (2), we fix a point $x \in S$ and assume that $\mathrm{St}_x S$ is good in $\mathrm{St}_x P$. The conical structure of stars implies that $\mathrm{Lk}_x S$ is good in $\mathrm{Lk}_x P$. Moreover, $\mathrm{St}_x P - \mathrm{St}_x S$ is connected by Proposition 1.3 and by deformation retraction $\mathrm{Lk}_x P - \mathrm{Lk}_x S$ is connected too. Therefore $\mathrm{Lk}_x P$ is connected, since $\mathrm{Lk}_x P - \mathrm{Lk}_x S$ is a dense subspace of it. Finally, (2) implies that $S \subset P$ is good, by Lemma 1.1 and Proposition 1.3. □

Next result is proven by induction on the dimension of P, using Proposition 1.4.

Proposition 1.5. *Let P be a polyhedron and $S \subset P$ be a subpolyhedron. If S is union of good subpolyhedra of P, then S is a good subpolyhedron of P.*

Branched coverings

Let $f : P \to Q$ be a non-degenerate PL map between polyhedra. A point $x \in P$ is *regular* for f if the restriction $f_| : \mathrm{St}_x P \to \mathrm{St}_{f(x)} Q$ is a homeomorphism, and it is *singular* otherwise. The *singular set* $S_f \subset P$, consisting of all the singular points for f, is a subpolyhedron of P. The subpolyhedron $B_f = f(S_f) \subset Q$ is called the *branch set* of f, and the subpolyhedron $S'_f = \mathrm{Cl}(f^{-1}(B_f) - S_f) \subset P$ is called the *pseudo-singular set* of f. Moreover, we put $T_f = S_f \cup S'_f = f^{-1}(B_f) \subset P$.

By a *branched covering* we mean a non-degenerate PL map $f : P \to Q$ between non-empty polyhedra, such that S_f is a good subpolyhedron of P, B_f is a good subpolyhedron of Q and Q is connected.

Our definition is different from Fox's one, since we don't require P to be connected and he requires T_f istead of S_f to be a good subpoliedron of P. However, this does not make a real difference (see page 250 of [7] and Proposition 1.8 below).

Proposition 1.6. *Let $f : P \to Q$ be a branched covering. Then f is a surjective open map and T_f (resp. B_f) has local codimension ≥ 2 in P (resp. Q).*

Proof. Since f is a closed map, while $f_| : P - S_f \to Q$ is an open map, then $f(P - T_f)$ is a non-empty open and closed subset of $Q - B_f$. By Proposition 1.3, $Q - B_f$ is connected. Then $f(P - T_f) = Q - B_f$, hence f is surjective. By the same argument, $f_| : \mathrm{St}_x P \to \mathrm{St}_{f(x)} Q$ is surjective for every $x \in P$. This proves that f is an open map.

Let K and L be triangulations respectively of P and Q with respect to which f is a simplicial map, and let $H \subset K$ the subcomplex such that $S_f = |H|$. We need to prove that any top simplex $\sigma \in H$ of dimension m is a face of an $(m+2)$-simplex of K. By contradiction, assume that all the top simplexes of K containing σ have dimension $m+1$. In fact, there is exactly one such top simplex of K, otherwise S_f locally disconnect P. But this easily implies that $\sigma \notin H$. □

Proposition 1.7. *Let $f : P \to Q$ be a non-degenerate PL map between polyhedra and assume that Q is connected. Then f is a branched covering if and only if one of the following equivalent properties holds:*
(1) $\mathrm{St}_x f = f_| : \mathrm{St}_x P \to \mathrm{St}_{f(x)} Q$ *is a branched covering for every* $x \in S_f$*;*
(2) $\mathrm{Lk}_x f = f_| : \mathrm{Lk}_x P \to \mathrm{Lk}_{f(x)} Q$ *is a connected branched covering for every* $x \in S_f$.

Proof. Given f as in the statement and $x \in P$, both $\mathrm{St}_x f$ and $\mathrm{Lk}_x f$ are non-degenerate PL maps. Then, the equivalence between (1) and (2) follows, by Proposition 1.4, from the conical structure of f at x. In particular, the connectedness of $\mathrm{Lk}_x P$ (resp. $\mathrm{Lk}_{f(x)} Q$) is related to the goodness of $S_{\mathrm{St}_x f}$ at x (resp. $B_{\mathrm{St}_x f}$ at $f(x)$).

On the other hand, a direct inspection shows that $\mathrm{St}_x S_f = S_{\mathrm{St}_x f}$ for every $x \in S_f$ and $\mathrm{St}_y B_f = \cup_{x \in f^{-1}(y)} B_{\mathrm{St}_x f}$ for every $y \in B_f$. Hence, by Propositions 1.2, 1.4 and 1.5, f is a branched covering if and only if it satisfies property (1). □

Proposition 1.8. *Let $f : P \to Q$ be a branched covering and $S \subset P$ be a subpolyhedron. Then S is a good subpolyhedron of P if and only if $f(S)$ is a good subpolyhedron of Q. In particular, T_f is a good subpolyhedron of P.*

Proof. Since both the notions of branched covering and good subpolyhedron are local in nature, it is enough to deal with the case where $\dim P = \dim Q$ is finite. We argue by induction on such dimension, starting from the trivial case of dimension 0.

For every $x \in S$, $\mathrm{Lk}_x f : \mathrm{Lk}_x P \to \mathrm{Lk}_{f(x)} Q$ is a branched covering between polyhedra of lower dimension (Proposition 1.7). Thus $\mathrm{Lk}_x S$ is good in $\mathrm{Lk}_x P$ if and only if $f(\mathrm{Lk}_x S)$ is good in $\mathrm{Lk}_{f(x)} Q$, by the inductive hypothesis. Moreover, $\mathrm{Lk}_x P$ is connected if and only if $\mathrm{Lk}_{f(x)} Q$ is connected. Then the thesis follows by Propositions 1.7, 1.4 and 1.5, since $\mathrm{Lk}_y f(S) = \cup_{x \in f^{-1}(y)} f(\mathrm{Lk}_x S)$ for every $y \in f(S)$. □

Given a branched covering $f : P \to Q$, the restriction $g = f_| : P - T_f \to Q - B_f$ is a PL ordinary covering. In fact, $\mathrm{St}_y Q$ is evenly covered for every $y \in Q - B_f$, by the definition of branch set. Since $Q - B_f$ is connected, all the fibers of g have the same cardinality $n \leq \infty$, that we call the *degree* of f, $d(f) = n$. We call *monodromy* of f the usual monodromy homomorphism $\omega_f : \pi_1(Q - B_f) \to \Sigma_{d(f)}$ of the covering g. We also define the *local degree* of f at $x \in P$ as $d_x(f) = d(\mathrm{St}_x f)$ and the monodromy $\omega_{f,y} : \pi_1(\mathrm{St}_y L - B_f) \to \Sigma_{d(f)}$ of f at $y \in Q$ as that of the restriction of f over $\mathrm{St}_y Q$. The local degree $d_x(f)$ is finite for every $x \in P$, due to the local compactness of polyhedra. Hence, the monodromy $\omega_{f,y}$ has finite orbits for every $y \in Q$. In fact, such orbits correspond to the restrictions $\mathrm{St}_x f$ with $x \in f^{-1}(y)$.

The covering space P and the branched covering f can be reconstructed from the other data, namely from Q and g or from Q and ω_f. This can be done by the following completion criterion provided by Fox in [7], whose proof is a simple restatement of Fox's results.

Proposition 1.9. *Let Q be a connected polyhedron, $B \subset Q$ be a good subpolyhedron and $g : R \to Q - B$ be a PL ordinary covering, whose monodromy $\omega_{g,y}$ at y has finite orbits for every $y \in B$. Then there exist a simplicial complex P, a good subpolyhedron $T \subset P$, a PL homeomorphism $h : P - T \to R$ and a branched covering $f : P \to Q$, uniquely determined up to PL homeomorphisms, such that $f_| : P - T \to Q - B$ coincides with $g \circ h$.*

According to Fox, the branched covering f is called the *completion* of the ordinary covering g over the complex Q.

The following elementary results will be useful to define branchfolds and deal with them. When the proof follows easily from the previous results, we omit it.

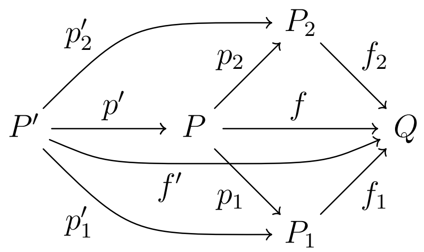
Proposition 1.10. *Let $f : P \to Q$, $g : Q \to R$ and $g \circ f : P \to R$ be PL maps between polyhedra and assume that Q is connected. If any two of the three maps f, g and $g \circ f$ are branched coverings, then the third one is a branched covering too.*

Corollary 1.11. *If $f : P \to Q$ and $g : Q \to R$ are branched coverings and g has finite degree, then the composition $g \circ f : P \to R$ is a branched covering.*

Given two branched coverings $f_i : P_i \to Q$ with P_i connected, $i = 1, 2$, we define their pullback as follows. We put $B = B_{f_1} \cup B_{f_2}$ and $R_i = P_i - f_i^{-1}(B)$, $i = 1, 2$. By Propositions 1.5 and 1.8, these are good subpolyhedra of Q and P_i respectively. Then, we consider the fiber product of $g_i = f_{i|} : R_i \to Q - B$, $i = 1, 2$, consisting of the polyhedron $R = \{(x_1, x_2) \in R_1 \times R_2 \mid g_1(x_1) = g_2(x_2)\}$ together with the projections $\pi_i : R \to R_i$, $i = 1, 2$. The maps π_i and $g = g_1 \circ \pi_1 = g_2 \circ \pi_2 : R \to Q - B$ are ordinary coverings. Let p_i and f be the corresponding completions: f is a branched covering of degree $d(f) = d(f_1)\, d(f_2)$, with branch set $B_f = B_{f_1} \cup B_{f_2}$ and monodromy $\omega_f = \omega_{f_1} \times \omega_{f_2} : \pi_1(Q - B_f) \to \Sigma_{d(f_1)} \times \Sigma_{d(f_2)} \subset \Sigma_{d(f_1)\, d(f_2)}$. By the uniqueness of completions p_i and f can be assumed to share the same covering space P. Hence, they fit into a commutative diagram of branched coverings (see diagram 1).

We call $f : P \to Q$ the *pullback* of the connected branched coverings $f_i : P_i \to Q$, $i = 1, 2$. In spite of the assumption that P_i is connected, P is not necessarily connected. Moreover, the diagram is not a pullback in the category of PL maps. However, according to next result, it is a pullback in the category of branched coverings (cf. [8] and [13] for some very special cases).

Proposition 1.12. *Let $f_i : P_i \to Q$, $i = 1, 2$, be connected branched coverings and $f : P \to Q$ be their pullback. If there exist branched coverings $p'_i : P' \to P_i$, $i = 1, 2$, such that $f' = f_1 \circ p'_1 = f_2 \circ p'_2$, then there exists a PL map $p' : P' \to P$ such that $f' = f \circ p'$. Moreover, p' restricts to a branched covering over each connected component of P.*



(1)

Proof. We consider the subpolyhedra $B = B_f \cup B_{f'} = B_{f_1} \cup B_{f_2} \cup B_{f'} \subset Q$, $R_i = P_i - f_i^{-1}(B) \subset P_i$, $R = P - f^{-1}(B) \subset P$ and $R' = P' - f'^{-1}(B) \subset P'$, which are good by Propositions 1.5 and 1.8, and the ordinary coverings $g_i = f_{i|} : R_i \to Q - B$, $g = f_| : R \to Q - B$ and $g' = f'_| : R' \to Q - B$. Since g' factorizes through g_1 and g_2, we have that $g'_*(\pi_1(R', x')) \leqslant g_{1*}(\pi_1(R_1, x_1)) \cap g_{2*}(\pi_1(R_2, x_2)) = g_*(\pi_1(R, x)) \leqslant \pi_1(Q - B, y)$, for any base points $x = (x_1, x_2) \in R$ and $x' \in R'$ such that $p'_i(x') = x_i$. This allows us to lift componentwise g' through g in order to get an ordinary covering $h : R' \to R = P - f^{-1}(B)$ such that $g' = g \circ h$. Then p' can be obtained as the completion of h over P. The uniqueness of completions and liftings gives the commutativity of the diagram, while the last part of the statement is true by construction. □

A branched covering $f : P \to Q$ is called *regular* if there is a group G of PL automorphisms of P and a PL homeomorphism $h : P/G \to Q$ such that $f = h \circ \pi_G$, where $\pi_G : P \to P/G$ is the canonical projection.

Given a polyhedron P and a group G of PL automorphisms of P, the projection $\pi_G : P \to P/G$ is a non-degenerate PL map onto a polyhedron P/G if and only if the action of G on P is properly discontinuous. This is equivalent to the existence of a triangulation $P = |K_G|$ which makes the action simplicial. If this is the case, π_G is simplicial with respect to K_G'' and its singular set S_{π_G} is a G-invariant subpolyhedron of P triangulated by a subcomplex of K_G''. By definition, S_{π_G} consists of all the points $x \in P$ whose stabilizer G_x is bigger than the stabilizer $G_{\mathrm{St}_x P}$ of $\mathrm{St}_x P$. In the following we will write S_G and B_G to indicate the corresponding subpolyhedra $S_{\pi_G} = T_{\pi_G}$ and B_{π_G} associated to the canonical projection π_G.

Definition 1.13. A *good action* of G on P is an effective properly discontinuous PL action of G on P, such that S_G is a good subpolyhedron of P.

Proposition 1.14. *Let P be a polyhedron with an effective PL action of a group G on it, such that P/G is connected. Then, the action is good if and only if the canonical projection $\pi_G : P \to P/G$ is a branched covering.*

Proof. The only non-trivial fact to be proved is that $\pi_G(S_G)$ is a good subpolyhedron of P/G when the action of G on P is good. In fact, since S_G is G-invariant, then $\mathrm{St}_{\pi_G(x)} P/G - \pi_G(S_G) = \pi_G(\mathrm{St}_x P - S_G)$ for every $x \in S_G$. Therefore the goodness of $\pi_G(S_G) \subset P/G$ follows from that of $S_G \subset P$, by using Proposition 1.1. □

If $f : P \to Q$ is a regular branched covering, then any restriction $f_| : A \to Q$ to a connected component A of P is a regular branched covering. Namely, if $f \cong \pi_G : P \to P/G$, then $f_| \cong \pi_H : A \to A/H$, where H is the group of PL automorphisms of A consisting of the restrictions of those $g \in G$ such that $g(A) = A$. Moreover, up to PL homeomorphisms, the branched covering $f_| : A \to Q$ does not depend on the choice the component A, since the subgroups of G leaving invariant different components of P are conjugate in G. So, it makes sense to call $f_| : A \to Q$ the *connected restriction* of the regular branched covering $f : P \to Q$. The rest if this subsection is then focused on branched coverings whose covering space is connected. We will call them *connected branched coverings*. By using completions, it can be easily shown that a connected branched covering $f : P \to Q$ is regular if and only if the associated ordinary covering $g = f_| : P - T_f \to Q - B_f$ is regular as well. Moreover, if this is the case and f is induced by the action of a group G on P, then the lifting properties of the connected ordinary covering g imply that the singular set $S_f = S_G$ consists of all the points $x \in P$ whose stabilizer G_x is non-trivial.

Proposition 1.15. *Let $f \cong \pi_G : P \to Q$ be the connected regular branched covering induced by a good action of G on P. Then the restriction $f_| : S \to T$ to connected open subspaces $S \subset P$ and $T \subset Q$ is a (connected regular) branched covering if and only if S is a connected component of $f^{-1}(T)$. In this case, $f_| \cong \pi_H$ is induced by the good action of $H = \{g \in G | g(S) = S\} \leqslant G$ on S given by restriction.*

Proof. The first claim follows by completion from the analogous property of connected ordinary coverings. The second claim follows from Proposition 1.14, as

$f_| : S \to T$ represents the connected restriction of $f_| : f^{-1}(T) \to T$ and the action of H on S is effective, since a deck transformation of a connected ordinary covering is uniquely determined by the image under it of any given base point. □

Proposition 1.16. *Let P be connected, with a good action of G on it. Then the restriction of the action to any $H \leqslant G$ is good and the canonical projection $\pi : P/H \to P/G$ is a branched covering, regular if and only if H is normal in G. On the other hand, a branched covering $f : P \to Q$ factorizing the canonical projection $\pi_G : P \to P/G$ is regular, being PL homeomorphic to $\pi_H : P \to P/H$ for $H \leqslant G$.*

Proof. Given $H \leqslant G$ as in the statement, we have that $S_H \subset S_G$. Then the restriction of the action of G to H is good (Proposition 1.2). Hence π is a branched covering (Propositions 1.10 and 1.14). The rest of the statement follows by completion, after noticing that the claimed facts hold for ordinary covering. □

By a *regularization* of the connected branched covering $f : P \to Q$ we mean any connected regular branched covering $r : R \to Q$ which factorizes as $r = f \circ s$ for some (regular) branched covering $s : R \to P$. We call r the *minimal regularization* of f if it also satisfies the universal property that, for any other connected regular branched covering $r' = f \circ s' : R' \to Q$, there exists a (regular) branched covering $t : R' \to R$ fitting into a commutative diagram (as a consequence, if such minimal regularizations exists, then it is uniquely determined up to PL homeomorphisms).

Proposition 1.17. *A connected branched covering $f : P \to Q$ has a regularization if and only if the monodromy $\omega_{f,y}$ is finite for every $y \in Q$. If this is the case, the minimal regularization of f exists as well, and its degree is finite if $d(f)$ is.*

Proof. Assume that f has a regularization $r = f \circ s : R \to Q$. Then, for $y \in Q$, the finiteness of $\omega_{f,y}$ follows from that of $\omega_{r,y}$, which is in turn deduced from that of $\omega_{\mathrm{St}_x r}$, $x \in r^{-1}(y)$ by a standard argument. Viceversa, assume that the monodromy $\omega_{f,y}$ is finite for every $y \in Q$. Then it can be proven that the connected restriction $r : R \to P$ of the completion over Q of the ordinary covering of $Q - B_f$ is a regularization of f. To prove the existence of the minimal regularization, consider the regular covering $o : O \to Q - B_f$ such that $M = \operatorname{Im} o_*$ is the largest normal subgroup of $\pi_1(Q - B_f)$ contained in $f_{|*}(\pi_1(P - T_f))$. Hence o can be completed to a regular branched covering $r_M : R_M \to Q$, which turns out to be the required minimal regolarization. Moreover, is f has finite degree d, then by construction its regularization has finite degree $\leq d!$, hence the same holds for the minimal one. □

Next proposition easily follows from Proposition 1.9.

Proposition 1.18. *Let $f_i : P_i \to Q$ $(i = 1, 2)$ be connected branched coverings and $f : P \to Q$ be their pullback. If f_1 (resp. f_2) is regular, induced by a good action of a group G_1 on P_1 (resp. G_2 on P_2), then p_2 (resp. p_1) is regular, induced by the lifting of the given action to a good action of the same group on P. On the other hand, if both f_i are regular as above, then $f : P \to Q$ is regular, induced by the natural good action of $G_1 \times G_2$ on P.*

As a consequence, if at least one of the connected branched coverings $f_i : P_i \to Q$ is regular, the restrictions of their pullback $f : P \to Q$ to the connected components of P are all equivalent up to PL homeomorphisms (the same holds for the projections p_i). We call any such restriction the *connected pullback* of f_1 and f_2: it can be thought as the composition of the connected restriction of the regular branched covering p_2 (resp. p_1) in diagram 1 with f_2 (resp. f_1), if f_1 (resp. f_2) is regular, while it coincides with the connected restriction of the pullback of f_1 and f_2 as branched coverings, if they both are regular. From a different perspective, the connected pullback of f_1 and f_2 is the completion of the ordinary covering $g : R \to Q - B$ such that $B = B_{f_1} \cup B_{f_2}$ and $g_*(\pi_1(R)) = f_{1|*}(\pi_1(P_1 - f_1^{-1}(B))) \cap f_{2|*}(\pi_1(P_2 - f_2^{-1}(B)))$.

Next result immediately follows from Propositions 1.12 and 1.18.

PROPOSITION 1.19. *Let $f_i : P_i \to Q$ ($i = 1, 2$) be connected branched coverings, at least one of which is regular, and let $f : P \to Q$ be their connected pullback. Then f_1, f_2 and f fit into a commutative diagram (see diagram 1) satisfying the universal pullback property in the category of the connected branched coverings.*

Pseudo-manifolds

Most of the polyhedra we will deal with in the next section, in particular branchfolds themselves, belong to a particular class of polyhedra called pseudo-manifolds.

An m-dimensional *pseudo-manifold* is a polyhedron P having a triangulation K with the following properties, which actually hold for any triangulation K of P:
(1) any top simplex of K has dimension m (P is homogeneously m-dimensional);
(2) any $(m-1)$-simplex of K is a face of exactly two m-simplexes of K;
(3) $P_{m-2} = |K_{m-2}|$ is a good subpolyhedron of P.

Moreover, P is said to be *orientable* if the top simplexes of K can be coherently oriented, and it is said to be *locally orientable* if this can be done for the top simplexes of $\mathrm{St}(x, K)$ for each vertex x of K. Such a coherent orientation is called an *orientation* (resp. a *local orientation*) of P. Of course, PL manifolds without boundary are locally orientable pseudo-manifolds. Viceversa, pseudo-manifolds are PL manifolds with certain local singularities in codimension > 2, being locally Euclidean at the points in the interior of all the simplexes of codimension ≤ 2. In particular, locally orientable orbifolds without boundary are locally orientable pseudo-manifolds.

Our definition of pseudo-manifold is not completely standard. Indeed, property (3) is usually replaced by the requirement that the connected components of P are strongly connected. By Proposition 1.3, this follows from property (3), but the viceversa does not hold. So, in this respect our notion of pseudo-manifold is more restrictive than the usual one. On the other hand, property (3) presents the advantage of being local in nature. Hence, according to our definition, any open subspace of a pseudo-manifold is a pseudo-manifold of the same dimension. Moreover, we have the following proposition, which follows immediately from Propositions 1.6 and 1.8.

PROPOSITION 1.20. *Let $f : P \to Q$ be a branched covering. Then P is a pseudo-manifold of dimension m if and only if Q is a pseudo-manifold of the same dimension m. In this case, orientability (resp. locally orientability) of Q lifts to P.*

As we already noticed, properties (1) to (3) of a pseudo-manifold P hold for any triangulation of P. As a consequence, a subpolyhedron $S \subset P$ is good if and only if it has local codimension ≥ 2. Indeed, property (2) implies that a good subcomplex cannot contain any codimension 1 simplex. Viceversa any subpolyhedron of codimension ≥ 2 is good by property (3) and Proposition 1.2, being contained into a codimension 2 skeleton. With this characterization, the previous subsections become simpler in the context of pseudo-manifolds. In particular, the following propositions rewrite the definition of branched covering and good action in such context.

Proposition 1.21. *Let $f : P \to Q$ be a non-degenerate PL map between pseudo-manifolds of dimension m and assume that Q is connected. Then f is a branched covering if and only if* $\dim S_p \leq m-2$ *and* $\dim B_p \leq m-2$.

Proposition 1.22. *Let P be a pseudo-manifold of dimension m. Then a properly discontinuous action of a group G on P is a good action if and only if* $\dim S_G \leq m-2$. *In particular, any properly discontinuous orientation preserving action on an orientable pseudo-manifold is good.*

2. Branchfolds

In this section we introduce the notion of branchfold and we prove some fundamental results about branchfolds spaces and related maps. We consider only branchfolds without boundary, but the extension to the bounded case is straightforward.

Branchfolds generalize locally orientable orbifolds without boundary, in that they admit a much wider class of singularities. These include codimension two cone singularities with any rational angle of $2k\pi/h$ radians, with h and k positive coprime integers, instead of only those of $2\pi/h$ radians allowed for orbifolds. The idea is to define an m-dimensional branchfold as a space covered by open sets U modelled by two regular branched coverings $V \leftarrow P \to U$ with V open in R^m, rather than only one $V \to U$ like in the orbifold case. The basic example is when all the spaces U, P and V coincide with the open disk D^2 and the two regular branched coverings are p_h and p_k, respectively induced by the cyclic actions on D^2 generated by the rotations of $2\pi/h$ and $2\pi/k$ radians, with h and k coprime positive integers. It is worth remarking that the two group actions modelling a branchfold chart are always required to generate a single effective action on P as in the above example, but in general this is not the direct (or even a semidirect) product of them.

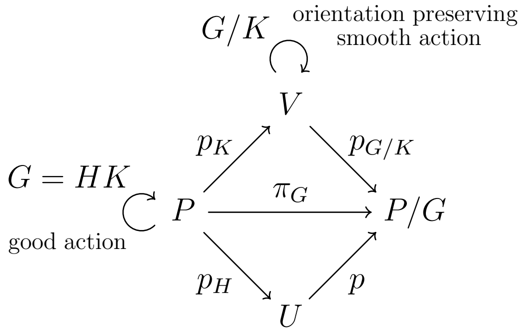
The setup of these local data, the branchfold charts, requires a long preparatory work, in order to see how they can be glued together to give a branchfold structure. In principle this is done in the same way as for orbifolds, but details are more complicated, as we have to take into account two group actions instead of only one.

Charts and structures

Let X be a polyhedron.

Definition 2.1. An *m-branchfold chart* on X is a sextuple $(U, \varphi, P, \psi, V, G)$, where: $U \subset X$ and $V \subset R^m$ are both open; P is a connected polyhedron; $G = HK$ is a finite group generated by $H \leqslant G$ and $K \trianglelefteq G$; a good action of G on P is given, such that $\varphi : U \to P/H$ is a PL homeomorphism and $\psi : P/K \to V$ is a piecewise regular smooth homeomorphism letting the induced action of G/K on P/K correspond to an orientation preserving smooth action on V.

We denote by π_G, π_H and π_K the canonical projections of the good action of G on P and of its restrictions to H and K, and by $\pi_{G/K}$ the canonical projection of the induced action of G/K on P/K (which exists since K is normal in G). By Proposition 1.16, all these maps are regular branched coverings, while π is a (possibly irregular) branched covering. In a branchfold chart as above we identify P/H with $U \subset X$ and P/K with $V \subset R^m$ respectively through φ and ψ. We also put $p_H = \varphi^{-1} \circ \pi_H$, $p_{G/K} = \pi_{G/K} \circ \psi^{-1}$, $p_K = \psi \circ \pi_K$ and $p = \pi \circ \varphi$, to get the commutative diagram below. Hence, we can omit φ and ψ and denote the branchfold chart by the quadruple $C = (U, P, V, G = HK)$, where $H \leqslant G$ and $K \trianglelefteq G$ are explicitly indicated.



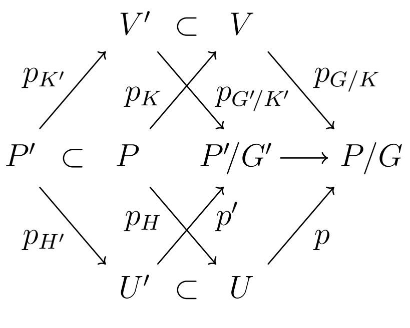
(2)

All the polyhedra in the diagram are orientable pseudo-manifolds. This is trivially true for V, while it follows from Proposition 1.20 for P. Moreover, since the action of G/K is orientation preserving, the action of G on P is orientation preserving as well. Hence, Proposition 1.20 implies that P/G and U are orientable pseudo-manifolds. Actually, $P/G \cong V/(G/K)$ is a locally orientable orbifold.

Two branchfold charts C_1 and C_2 are called *isomorphic* if there is an isomorphism $\eta : G_1 \to G_2$, such that $\eta(H_1) = H_2$ and $\eta(K_1) = K_2$, and an η-equivariant PL homeomorphism $h : P_1 \to P_2$ inducing a diffeomorphism $V_1 \cong V_2$. Moreover, they are called *strongly isomorphic* if $U_1 = U_2$ and the PL homeomorphism $U_1 \cong U_2$ induced by h is the identity.

Definition 2.2. An m-branchfold chart C' is called a *restriction* of the m-branchfold chart C if $p_{H'}$ and $p_{K'}$ are restrictions of p_H and p_K respectively to the open subspaces $P' \subset P$, $U' \subset U$ and $V' \subset V$.

By Propositions 1.15 and 1.16, $H' \leqslant H$, $K' \trianglelefteq K$ and the action of $G' \leqslant G$ on P' is the restriction of that of G on P. More precisely, $H' = \{h \in H \mid h(P') = P'\} \leqslant H$ and $K' = \{k \in K \mid k(P') = P'\} \leqslant K$. The diagram below, where $P'/G' \to P/G$ is the composition of the inclusion $P'/G' \subset P/G'$ with the canonical projection $P/G' \to P/G$, summarizes how the restriction chart is related to the original one.



(3)

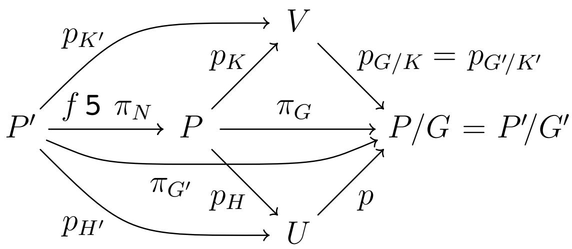
Of course, chart restriction is a transitive and anti-symmetric binary relation, hence it induces a partial order on the set of all branchfold charts. A chart restriction as above always exists with U' an arbitrarily small open neighborhood of any given

$x \in U$. When considering local models, we will see that such a chart can be chosen to be very special. Here we limit ourselves to derive its existence from the fact that the map $p : U \to P/G$ in diagram 2 is a spread in the sense of Fox [7], i.e. the connected components of the counterimages under p of the open subsets of P/G form a basis for the topology of U. Thus, we can find a connected open subset $W \subset P/G$ such that $p(x) \in W$ and the connected component U' of $p^{-1}(W)$ containing x is arbitrarily small. Then, we define C' by setting P' to be any connected component of $p_H^{-1}(U')$, $V' = p_K(P')$,$H' = \{h \in H \mid h(P') = P'\}$ and $K' = \{k \in K \mid k(P') = P'\}$. This is a branchfold chart by Proposition 1.15. The restriction chart we constructed is not a generic one. Indeed, being P' a connected component of $\pi_G^{-1}(W)$, $\pi_{G|} : P' \to \pi_G(P') = W$ is a regular branched covering. Hence the map $P'/G' \to P/G$ can be thought as the composition of a regular branched covering $P'/G' \to \pi_G(P')$ with the inclusion $\pi_G(P') \subset P/G$. We will refer to such a restriction as a *special restriction*.

Now, we want to consider equivalent two m-branchfold charts when they give isomorphic local singularities on the base space X, independently on the specific polyhedra and good actions used to describe them. This equivalence relation is generated by that of domination defined here below.

Definition 2.3. An m-branchfold chart $(U, P', V, G' = H'K')$ is said to *dominate* the m-branchfold chart $(U, P, V, G = HK)$, which is called a *reduction* of the first, if there exists a PL map $f : P' \to P$ such that $p_{H'} = p_H \circ f$ and $p_{K'} = p_K \circ f$.

In this case f is a regular branched covering. Namely, Propositions 1.10 and 1.16 apply to the factorizations $p_{K'} = p_K \circ f$ and $p_{H'} = p_H \circ f$, providing a normal subgroup $N \trianglelefteq G'$ contained in $H' \cap K'$, such that $f \cong \pi_N : P' \to P'/N \cong P$, $G'/N \cong G$. Up to these isomorphisms, we have the following commutative diagram.



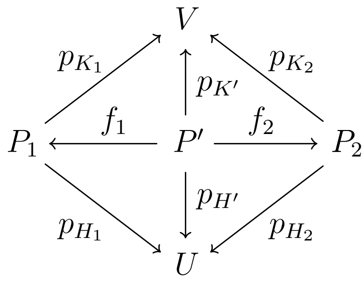
(4)

Viceversa, any normal subgroup $N \trianglelefteq G'$ contained in $H' \cap K'$ gives raise in this way to domination of $(U, P', V, G' = H'K')$ on $(U, P \cong P'/N, V, G = HK \cong H'/N\, K'/N \cong G'/N)$ through the map $f \cong \pi_N$.

Domination is a transitive and anti-symmetric binary relation, hence it induces a partial order on the set of all branchfold charts. Moreover, any special restriction of a domination (resp. reduction) chart is a domination (resp. reduction) of a special restriction. This is is not true if we do not require the restrictions to be special.

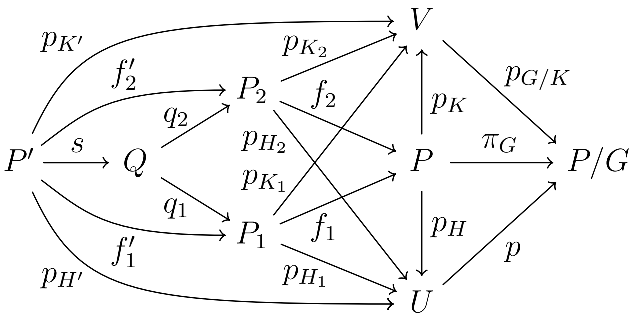
Definition 2.4. Two m-branchfold charts $(U, P_1, V, G_1 = H_1K_1)$ and $(U, P_2, V, G_2 = H_2K_2)$ are called *equivalent* if there exists a third m-branchfold chart $(U, P', V, G' = H'K')$ dominating both, as in the commutative diagram 5.

In order to see that chart equivalence is a true equivalence relation, the only non-trivial property to be verified is transitivity. Since chart domination is a transitive relation, it suffices to show that for any two charts $(U, P_1, V, G_1 = H_1K_1)$ and



(5)

$(U, P_2, V, G_2 = H_2K_2)$ dominating the same chart $(U, P, V, G = HK)$ there exists a chart $(U, P', V, G' = H'K')$ which dominates both of them. This is done in the commutative diagram 6. We start with the regular branched coverings $f_1 : P_1 \to P$ and $f_2 : P_2 \to P$ giving the assumed dominations. Then we consider their connected pullback $q = f_1 \circ q_1 = f_2 \circ q_2 : Q \to P$ and the minimal regularization $r = \pi_G \circ q \circ s : P' \to P/G$ of the composition $\pi_G \circ q$. The PL maps $p_{H'}$, $p_{K'}$, f_1' and f_2' are defined by composition. Proposition 1.17 allows us to think of r as the canonical projection $\pi_{G'}$ of a good action of a finite group G' on P'. By Proposition 1.16, $p_{H'}$ and $p_{K'}$ are regular branched coverings corresponding to the restrictions of that action to certain subgroups $H', K' \leqslant G'$. Moreover K' is normal in G', since $p_{G/K}$ is regular. On the other hand, it is clear from the diagram that $G' = H'K'$, hence we have a branchfold chart $(U, P', V, G' = H'K')$. As desired, this chart dominates $(U, P_1, V, G_1 = H_1K_1)$ and $(U, P_2, V, G_2 = H_2K_2)$ respectively through $f_1' : P' \to P_1$ and $f_2' : P' \to P_2$.



(6)

Now we can proceed with our main definitions.

Definition 2.5. An *m-branchfold atlas* on X is a set $\mathcal{A} = \{C_i\}_{i \in I}$ of m-branchfold charts such that $\mathcal{U} = \{U_i\}_{i \in I}$ is an open covering of X and the following *compatibility condition* holds: for any $i, j \in I$, $x \in U_i \cap U_j$, $\widetilde{x}_i \in P_i$ and $\widetilde{x}_j \in P_j$ such that $p_{H_i}(\widetilde{x}_i) = p_{H_j}(\widetilde{x}_j) = x$, there exist two restrictions $(U, P_i', V_i', G_i' = H_i'K_i')$ and $(U, P_j', V_j', G_j' = H_j'K_j')$ of the charts C_i and C_j respectively, with $x \in U$, $\widetilde{x}_i \in P_i'$ and $\widetilde{x}_j \in P_j'$, which are equivalent up to strong isomorphisms of charts.

The compatibility condition is summarized in the commutative diagram 7, where C' is the dominating branchfold chart providing the equivalence between the restrictions, $V_i' \cong V' \cong V_j'$ are diffeomorphisms and f_i and f_j are the domination maps.

Definition 2.6. An *m-branchfold* is a pair $X_\mathcal{B} = (X, \mathcal{B})$, where X is a polyhedron and $\mathcal{B}$ is an *m-branchfold structure*, i.e. a maximal m-branchfold atlas, on X. We will write X in place of $X_\mathcal{B}$ when no confusion can arise.

A standard argument shows that any m-branchfold atlas determines a unique m-branchfold structure containing it. In fact, the compatibility condition at a fixed

$$\begin{array}{ccccccccc}
V_i & \supset & V'_i & 5 & V' & 5 & V'_j & \subset & V_j \\
\uparrow p_{K_i} & & \uparrow p_{K'_i} & & \uparrow p_{K'} & & \uparrow p_{K'_j} & & \uparrow p_{K_j} \\
P_i & \supset & P'_i & \xleftarrow{f_i} & P' & \xrightarrow{f_j} & P'_j & \subset & P_j \\
 & \searrow p_{H_i} & & p_{H'_i} \searrow & \downarrow p_{H'} & \swarrow p_{H'_j} & & \swarrow p_{H_j} & \\
 & & U_i & \supset & U' & \subset & U_j & &
\end{array} \qquad (7)$$

$x \in X$ turns out to be an equivalence relation on the charts C such that $x \in U$. This will become evident in the next subsection, after expressing the compatibility condition in terms of local models.

Like locally orientable m-orbifolds without boundary, m-branchfolds are particular m-dimensional pseudo-manifolds. This follows from Proposition 1.20 and from the local nature of our notion of pseudo-manifold. On the other hand, locally orientable m-orbifolds without boundary coincide with the special m-branchfolds which admit a branchfold atlas $\mathcal{A} = \{C_i\}_{i \in I}$ such that $K_i = 1$ for every $i \in I$. Indeed, such branchfold charts reduce to usual orientable orbifold charts and the same is true for the compatibility condition between any two of them (cf. [2], [12], [16] or [17]).

Finally, we say that a branchfold is *orientable* (resp. *oriented*) referring to the underlying pseudo-manifold. Moreover, given an oriented branchfold X, by an oriented chart (resp. atlas) on X we mean a chart which is (resp. an atlas whose all charts are) oriented coherently with X.

Local models

The local model of a branchfold X at a point $x \in X$ can be characterized in terms of the conical branchfold charts centered at x which are minimal with respect to the domination order. In fact, such charts turn out to be all isomorphic. To prove this, we first formalize the definition of conical branchfold chart and show that all the conical restrictions at x of a given branchfold chart are isomorphic. Then, we show that each equivalence class of branchfold charts contains a unique representative that is minimal in the above sense. Finally, we put these facts together to get the claimed unicity up to isomorphism of the minimal conical charts at x.

Definition 2.7. A branchfold chart C of an branchfold X is called *conical* if P is an open cone, the action of G on P is conical (i.e. it preserves the cone structure of P) and the induced cone structure on V is linear. Moreover, we say that the chart is *centered at* x, when $x \in X$ is the apex of the cone structure induced on U.

Up to chart isomorphisms, in a conical chart we can always assume $V = R^m$ with apex at the origin, since any starlike open set is diffeomorphic to R^m, and $G/K < \mathrm{SO}(m)$ acting on R^m by Euclidean isometries, being G/K identifiable with a finite group of orientation preserving linear isomorphisms of R^m.

Given a branchfold chart C and a point $x \in U$, we can construct arbitrarily small branchfold conical restrictions C' of C centered at x as follows. Let $\widetilde{x} \in P$ be any point such that $p_H(\widetilde{x}) = x$ and let $\bar{x} = p_K(\widetilde{x}) \in V$. Consider (arbitrarily small) open stars $U' = \mathrm{St}_x U - \mathrm{Lk}_x U \subset U$, $P' = \mathrm{St}_{\widetilde{x}} P - \mathrm{Lk}_{\widetilde{x}} P \subset P$

and $V' = \mathrm{St}_{\widetilde{x}}V - \mathrm{Lk}_{\widetilde{x}}V \subset V$, and the stabilizer $G_{\widetilde{x}} \leqslant G$. Moreover, put $H' = H_{\widetilde{x}} = H \cap G_{\widetilde{x}} \leqslant H$, $K' = K_{\widetilde{x}} = K \cap G_{\widetilde{x}} \leqslant K$ and $G' = H'K' = H_{\widetilde{x}}K_{\widetilde{x}} \leqslant G_{\widetilde{x}} \leqslant G$. By Proposition 1.15, C' is a branchfold chart, hence a conical restriction of C.

In general G' is a proper subgroup $G_{\widetilde{x}}$. The simplest chart C where this happen is depicted in Figure 1. Here, $H = \langle\sigma\rangle \cong \mathbb{Z}_2$, $K = \langle\rho\rangle \cong \mathbb{Z}_2$ and $G = \langle\sigma, \rho\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, while the heavy lines represent cone singularities, whose angle is 2π divided by the corresponding numerical label. Then, for any $x \neq 0$ along the vertical axis of $U \cong R^3$ and any $\widetilde{x}$ such that $p_H(\widetilde{x}) = x$, we have $G' \cong 1$ while $G_{\widetilde{x}} = \langle\sigma\rho\rangle \cong \mathbb{Z}_2$.

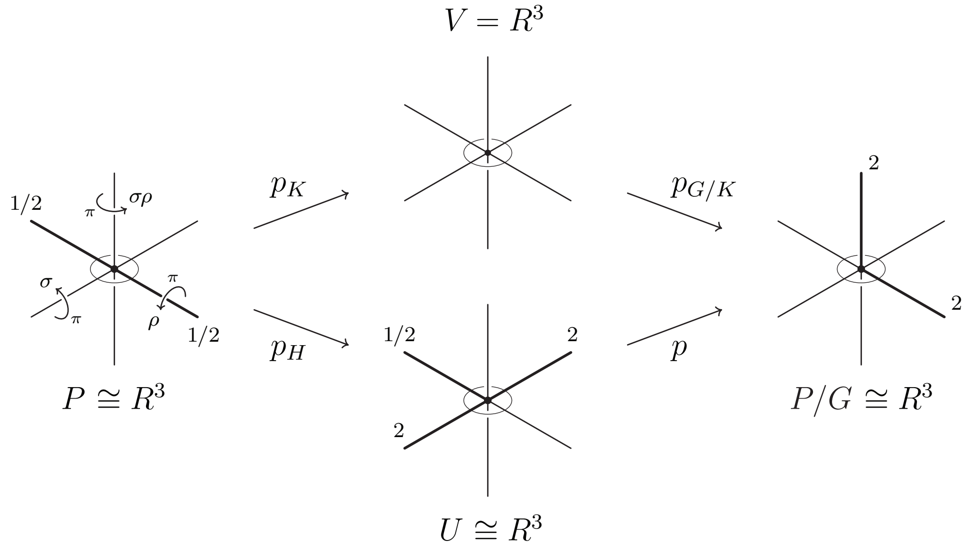


FIGURE 1.

All the conical restrictions of C centered at x are isomorphic. In fact, any such restriction can be obtained by the above construction, and this is independent on the choice of the lifting $\widetilde{x}$, up to conjugation by an element of H, and on the choice of the specific realizations of the stars, up to pseudo-radial PL homeomorphisms.

DEFINITION 2.8. A branchfold chart C is called *reduced* if it does not properly dominate any other chart (i.e. it is minimal with respect to the domination order).

The discussion following Definition 2.3 implies that C is reduced if and only if $H \cap K$ does not contain any non-trivial normal subgroup of G. Moreover, any branchfold chart C dominates a unique (up to isomorphisms) reduced chart $(U, P', V, G' = H'K')$, given by $P' = P/N$, $G' = G/N$, $H' = H/N$ and $K' = K/N$, where $N \trianglelefteq G$ is the maximal normal subgroup of G contained in $H \cap K$. Hence, any equivalence class of branchfold charts has a unique reduced representative (up to isomorphisms), as two equivalent reduced charts are dominated by the same chart.

PROPOSITION 2.9. *Let X be an branchfold. Then for any point $x \in X$ there exists a unique (up to isomorphisms) reduced conical chart $C_x = (U_x, P_x, V_x, G_x = H_xK_x$ of X centered at x.*

Proof. We know that conical charts of X centered at x do exist. The existence of a reduced one follows from the fact that any reduction of a conical chart is conical. The proof of the unicity follows from a standard argument, and we omit it. □

Definition 2.10. A branchfold *local model* is any reduced conical branchfold chart. Given a branchfold X and a point $x \in X$, we define the *local model* of X at x to be the reduced conical chart C_x whose existence and unicity are ensured by the previous proposition. Moreover, we call $G_x = H_x K_x$ the *isotropy group* of X at x and $i_x = |H_x|/|K_x|$ (where $|\cdot|$ denotes the cardinality) the *index* of X at x.

The restrictions and the intermediate chart dominating them in the compatibility condition for C_i and C_j in Definition 2.5 can be assumed to be conical. Hence those charts satisfy the compatibility condition at $x \in U_i \cap U_j$ if and only if their conical restrictions centered at x reduce to the same local model C_x up to isomorphism.

In terms of isotropy groups, locally orientable orbifolds without boundary can be characterized as the branchfolds X such that $K_x \cong 1$ for every $x \in X$. If this is the case $G_x = H_x$ and $G_x/K_x \cong G_x$. Moreover, there are isomorphisms $p_{K_x} : P_x \cong V_x \subset R^m$ and $p_x : U_x \cong P_x/G_x$ allowing us to identify p_{H_x} with p_{G_x/K_x}, which is always an orbifold local model, even for $K_x \ncong 1$. At the opposite end of the branchfold spectrum, we call X a *pure branchfold* when $H_x \subset K_x$ for every $x \in X$. In this case $G_x = K_x$ and $G_x/K_x \cong 1$, hence we have an isomorphism $p_{G_x/K_x} : V_x \cong P_x/G_x$. Up to this isomorphism, $p_{K_x} \cong p_x \circ p_{H_x}$ is the minimal regularization of the branched covering $p_x : U_x \to P_x/G_x \cong V_x$. Therefore, X is locally modelled on (conical) branched coverings of R^m. The simplest non-trivial example of such a local model is depicted in Figure 2, where $H = \langle\sigma\rangle \cong \mathbb{Z}_2$ and $G = K = \langle\sigma, \rho\rangle \cong \Sigma_3$, while p is the cone of the covering $S^2 \to S^2$ branched over three points with monodromies (1 2), (2 3) and (1 2 3) respectively.

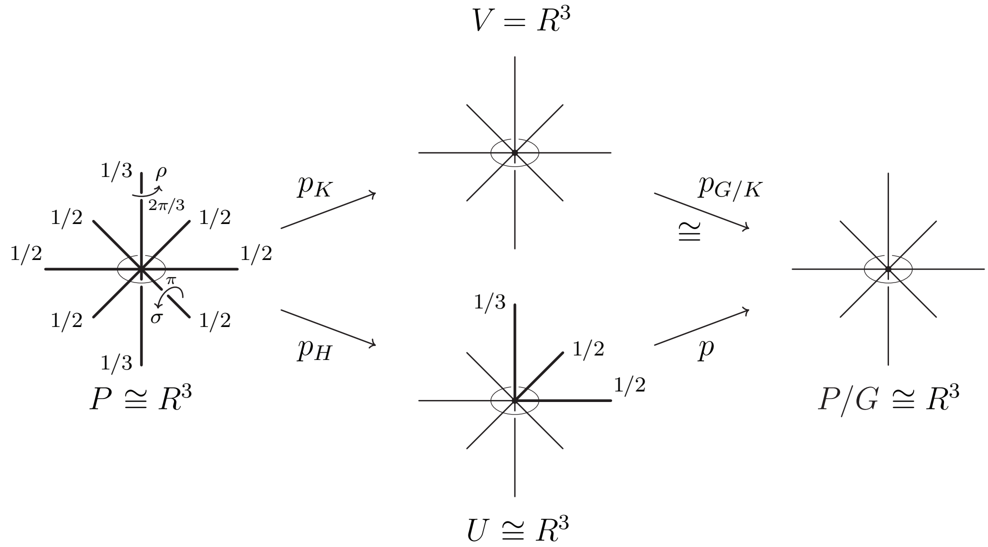


Figure 2.

Definition 2.11. Given an m-branchfold X, we define the *singular locus* of X to be the good subpolyhedron $\Sigma X = \{x \in X \mid G_x \neq 1\} \subset X$. We think of ΣX as a stratified set of dimension $\leq m - 2$, with the natural stratification such that the local model is constant on the connected components of the strata.

The fact that ΣX is good in X immediately follows from Propositions 1.2, 1.4 and 1.8, as $\mathrm{St}_x \Sigma X \subset p_{H_x}(S_{G_x})$ for every $x \in \Sigma X$. The natural stratification

$\Sigma_0 X \subset \Sigma_1 X \subset \ldots \subset \Sigma_{m-2} X = \Sigma X$ can be obtained by defining $\Sigma_i X$ as the set of all $x \in \Sigma X$ such that, for any triangulation of P_x making the action of G_x simplicial in the local model C_x, the simplex σ of the induced triangulation of U_x containing x in its interior has $\dim \sigma \leq i$. Considering the product structure of the chart over a neighborhood of the interior of σ, it is straightforward to verify that in this way we get a stratification. Moreover, such a product structure also implies that the local model is the same for all points in the interior of σ, hence this is contained in $\Sigma_i X$. For $x \in \Sigma_i X - \Sigma_{i-1} X$, there exists a simplex σ as above with $\dim \sigma = i$, which is a top simplex of $\Sigma_i X$. Then, the local model turns out to be locally constant on the stratum $\Sigma_i X - \Sigma_{i-1} X$, so it is constant on each connected component of it.

The next proposition focuses on the $(m-2)$-stratum $\Sigma_{m-2} X - \Sigma_{m-3} X$ of ΣX. Its proof is almost standard, and we leave it to the reader.

PROPOSITION 2.12. *Let X be an m-branchfold and let $x \in \Sigma X$ be a singular point in the $(m-2)$-stratum of ΣX. Then the local model C_x is given by $U_x \cong P_x \cong V_x \cong R^m$, $H_x = \langle \rho_{2\pi/h_x} \rangle \cong \mathbb{Z}_{h_x}$, $K_x = \langle \rho_{2\pi/k_x} \rangle \cong \mathbb{Z}_{k_x}$ and $G_x = \langle \rho_{2\pi/(h_x k_x)} \rangle \cong H_x \times K_x \cong \mathbb{Z}_{h_x k_x}$, where h_x and k_x are the unique coprime positive integers such that $i_x = h_x/k_x$, while ρ_α denotes the rotation of α radians around $R^{m-2} \subset R^m$.*

In the light of the previous proposition, for any connected component C of the $(m-2)$-stratum of ΣX, the branchfold structure of a neighborhood of C is completely determined by its index i_C, defined as the common index i_x at all the points $x \in C$. So, it makes sense to label each such component C with i_C, as we have done in Figures 1 and 2. These labels have integer values if X is an m-orbifold, and in this case they coincide with the customary ones, while they have values of the type $1/n$ if X is a pure m-branchfold. The viceversa does not hold in general, but it trivially holds when $\Sigma_{m-3} = \emptyset$, that is when the underlying space of X is a PL m-manifold and ΣX is a PL locally flat $(m-2)$-submanifolds of X.

Maps and coverings

This subsection is entirely devoted to introduce branchfold maps and coverings and to relate them to branched coverings.

DEFINITION 2.13. Let $f : X \to Y$ be a map between branchfolds. We call f a *branchfold map* if for every $x \in X$ there exist a branchfold chart C of X with $x \in U$, a branchfold chart C' of Y with $f(U) \subset U'$, a PL map $\varphi : P \to P'$ and a regular smooth map $\psi : V \to V'$, such that the following diagram commutes. Moreover, we call f a *branchfold isomorphism* if in addition it is a PL homeomorphism.

$$\begin{array}{ccc} V & \xrightarrow{\ \psi\ } & V' \\ {\scriptstyle p_K}\uparrow & & \uparrow{\scriptstyle p_{K'}} \\ P & \xrightarrow{\ \varphi\ } & P' \\ {\scriptstyle p_H}\downarrow & & \downarrow{\scriptstyle p_{H'}} \\ U & \xrightarrow{\ f_{|}\ } & U' \end{array} \tag{8}$$

The definition of branchfold map is of local nature. In particular, branchfold maps are locally PL, since so is $f_|$, but they are not necessarily PL. We also notice that the only reason why the composition of two branchfold maps could not be a branchfold map is the possible lack of regularity of the map ψ in the diagram.

By considering conical restrictions of both the branchfold charts involved in diagram 8, we can assume that $\psi : V \to V'$ is a linear map of maximum rank between linear spaces and $f_|$ is a PL map. Actually, C' could be assumed to be a local model, but in general one could not insist that C is a local model as well.

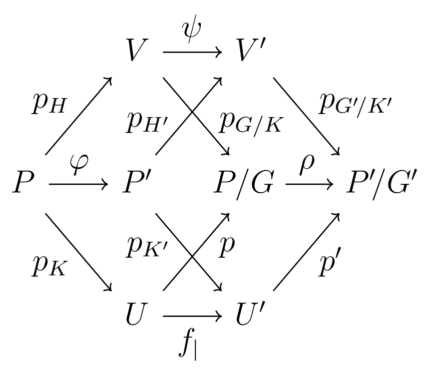
If $\dim X = \dim Y$, then the map $\psi : V \to V'$ can be assumed to be an isomorphism. In this case, Propositions 1.10 and 1.16 imply that φ is a regular branched covering, and then Proposition 1.10 and Corollary 1.11 imply that $f_|$ is a branched covering. Moreover, there exists a further branched covering $\rho : P/G \to P'/G'$ which completes diagram 8 to give the commutative diagram 9. Finally, by completion and the lifting properties of ordinary coverings, we can show that there exists a group homomorphism $\eta : G \to G'$ such that φ is η-equivariant, $\eta(H) \subset H'$ and $\eta(K) \subset K'$.

If $f : X \to Y$ is a branchfold isomorphism, then $\dim X = \dim Y$ and the map $f_|$ in diagram 9 is a PL homeomorphism. As a consequence, such diagram gives a domination of C on C', hence these branchfold charts are equivalent. Thus, branchfold isomorphisms can be characterized as those PL homeomorphisms that induce isomorphisms of local models at all points.

In order to give the definition of branchfold covering, we now need the concept of rectifiability of paths in finite dimensional polyhedra.

Let P be a finite dimensional polyhedron. Once a locally PL inclusion $P \subset R^n$ is given, we can define the length $L(\alpha)$ of a path $\alpha : [0,1] \to P$ in the usual way. $L(\alpha)$ depends on the inclusion, but the intrinsic metrics induced by different inclusions are locally Lipschitz equivalent. Hence the compactness of $\alpha([0,1])$ implies that the property $L(\alpha) < \infty$ does not depend on the inclusion. We say that the path $\alpha : [0,1] \to P$ is *rectifiable*, or that it has *finite length*, when $L(\alpha) < \infty$, for any locally PL inclusion $P \subset R^n$. If M is a smooth manifold, then the usual notion of rectifiability with respect to any Riemannian metric on M coincides with the above stated one, when M is thought with the polyhedral structure induced by any smooth triangulation of it.

DEFINITION 2.14. A *branchfold covering* is a branchfold map $f : X \to Y$ between branchfolds of the same dimension, which is *complete* with respect to lifting of rectifiable paths, meaning that any partial lifting $\widetilde{\alpha}_| : [0,1[\, \to X$ of a rectifiable path $\alpha : [0,1] \to Y$ extends to a complete lifting $\widetilde{\alpha} : [0,1] \to X$.



(9)

We remark that a branchfold covering $f : X \to Y$ is locally a branched covering at every point $x \in X$. Moreover, extending to the locally PL map f the notions of singular and branch set we have that $S_f \subset \Sigma X \cup f^{-1}(\Sigma Y)$ and $B_f \subset f(\Sigma X) \cup \Sigma Y$.

We emphasize that f it is not required to be a PL map, hence it is not necessarily a branched covering. Actually, S_f turns out to be always a good subcomplex of X, while B_f can be a quite pathological subset of Y, for example it can be dense in Y. However, branchfold coverings behave better than branched covering with respect to composition, being the composition of branchfold coverings always a branchfold covering. Arguing on diagram 8 for the local models of a branchfold covering $f : X \to Y$, we can easily realize that Y is an orbifold if X is and that X is a pure branchfold if Y is. Moreover, in the orbifold case, our definition of branchfold covering restricts to the usual one of orbifold covering, since the map φ in diagram 8 is forced to be a PL homeomorphism. We also note that in this case $f(\Sigma X) \subset \Sigma Y$, hence there is no room for pathological branch sets.

The next propositions relate PL branchfold coverings and branched coverings. Proposition 2.16 says that any branched covering $f : X \to Y$ of a given brachfold Y can be interpreted as a branchfold covering, by a suitable choice of the branchfold structure on X. An analogous result holds in the context of orbifolds only for proper maps, if one allows both the orbifold structures on X and Y to be suitably choosen. On the other hand, Proposition 2.17 extends the construction of good orbifolds as quotients of smooth manifolds by the action of a discrete group of diffeomorphisms.

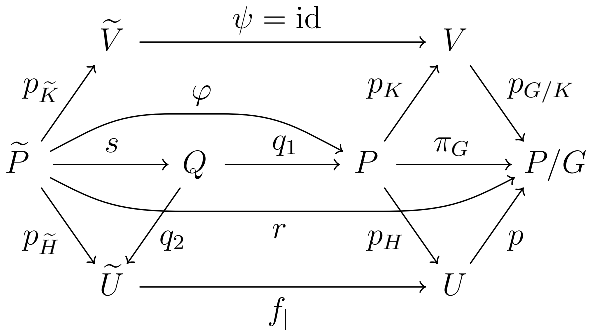
Proposition 2.15. *Let X and Y be branchfolds of the same dimension, and assume that Y is connected. Then, any PL branchfold map $f : X \to Y$ is a branchfold covering and it is a branched covering as a map between polyhedra.*

Proof. The local properties of branchfold maps between branchfold of the same dimension ensure that f is non-degenerate. Then, the first part of the statement derives from the observation that any non-degenerate PL map is complete, while the second part immediately follows from Proposition 1.7. □

Proposition 2.16. *Let $f : X \to Y$ be a branched covering onto a branchfold Y. Then, the branchfold structure of Y can be lifted to a unique branchfold structure on X turning f into a PL branchfold covering.*

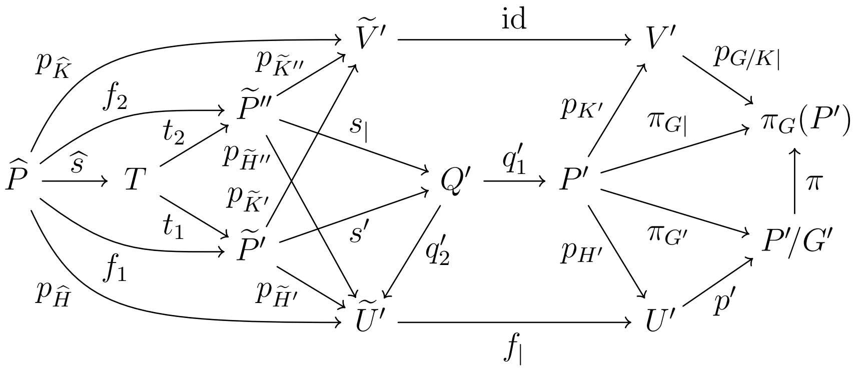
Proof. We first prove that any given branchfold chart C on Y can be lifted to a branchfold chart $\widetilde{C}$ on X, such that the two charts are related by a commutative diagram like diagram 8. This is done in diagram 10, which is constructed in the following way. Start with the diagram of the chart C on the right side of diagram 10. Choose an arbitrary connected component $\widetilde{U}$ of $f^{-1}(U)$ and consider the connected pullback $p_H \circ q_1 = f_| \circ q_2 : Q \to U$ of p_H and $f_| : \widetilde{U} \to U$. Then, let $r = \pi_G \circ q_1 \circ s$ be the minimal regularization of the branched coverings $\pi_G \circ q_1$. Finally, put $\widetilde{V} = V$ and define the remaining maps by composition. We denote by L the group of deck transformations of the covering r and define $\widetilde{H} \leqslant L$ and $\widetilde{K} \trianglelefteq L$ to be the subgroups corresponding to the coverings $p_{\widetilde{H}}$ and $p_{\widetilde{K}}$ respectively (see Proposition 1.16). Then, we put $\widetilde{G} = \widetilde{H}\widetilde{K} \leqslant L$, getting in this way a branchfold chart $\widetilde{C}$ on X, which is related to the original chart C on Y by the maps $f_|$, φ and $\psi = \mathrm{id}$, as in diagram 8.

Now, we have to verify that the lifted charts satisfy the compatibility condition of Definition 2.5, in such a way that they form a branchfold atlas (they obviously



(10)

cover all of X). It suffices to show that the local model $\widetilde{C}_x$, obtained from the lifted chart $\widetilde{C}$ by reduction of the conical restriction centered at any $x \in \widetilde{U}$, is equivalent to the lifting $\widetilde{C}_y$ of the local model C_y of Y at $y = f(x) \in U$. To this end, we first consider the conical restriction C' of the chart C centered at y and the lifting $\widetilde{C}'$ of it such that $x \in \widetilde{U}'$. We want to prove that such lifting is chart equivalent to the conical restriction $\widetilde{C}'$ of $\widetilde{C}$, where $\widetilde{P}''$ is any connected component of $\varphi^{-1}(P')$ such that $p_{\widetilde{H}}(\widetilde{P}'') = \widetilde{U}'$. The construction of $\widetilde{C}'$ is described by a diagram obtained from diagram 10 putting a prime on spaces and maps. We refer to this new diagram as diagram $10'$, without drawing it. Comparing diagrams 10 and $10'$, we see that there are open inclusions $U' \subset U$, $P' \subset P$, $V' \subset V$, $\widetilde{U}' \subset \widetilde{U}$ and $Q' \subset Q$, the first three by hypothesis and the last two by construction. Moreover, the maps $p_{H'}$, $p_{K'}$, $f_|$, q_1' and q_2' in diagram $10'$ are restrictions of the corresponding maps in diagram 10.



(11)

Then, we can construct the commutative diagram 11. Here, $s_|$, $\pi_{G|}$ and $p_{G/K|}$ are restrictions of maps in diagram 10. In particular, $s_|$ is a regular branched covering by Proposition 1.15, while $\pi_{G|}$ is a (finite) regular branched covering if we assume that the conical restriction C' is special. Hence, $p_{G/K|}$ is a regular branched covering by Proposition 1.15. Moreover, $t = s' \circ t_1 = s_| \circ t_2 : T \to Q'$ is the connected pullback of s' and $s_|$, while $\widehat{r} = \pi_G \circ q_1' \circ t \circ \widehat{s} : \widehat{P} \to \pi_G(P')$ is the minimal regularization of $\pi_G \circ q_1' \circ t$. Finally, the maps $p_{\widehat{H}}$, $p_{\widehat{H}}$, f_1 and f_2 are defined by composition. By putting $\widehat{G} = \widehat{H}\widehat{K}$, where $\widehat{H}$ and $\widehat{K}$ are the subgroups of the group of deck transformations of r corresponding to $p_{\widehat{H}}$ and $p_{\widehat{K}}$ respectively, we get a new chart $(\widetilde{U}', \widehat{P}, \widetilde{V}', \widehat{G} = \widehat{H}\widehat{K})$ on X. Looking at diagram 11, it is straightforward to verify that this chart dominates both $(\widetilde{U}', \widetilde{P}', \widetilde{V}', \widetilde{G}' = \widetilde{H}'\widetilde{K}')$ and $(\widetilde{U}', \widetilde{P}'', \widetilde{V}', \widetilde{G}'' = \widetilde{H}''\widetilde{K}'')$ through the maps f_1 and f_2 respectively, giving us the claimed equivalence between these two charts.

Since $\widetilde{C}_x$ is a reduction of $\widetilde{C}'$ by its very definition, we can conclude our argument by showing that the reduction of C' to C_y lifts to a reduction of $\widetilde{C}'$ to $\widetilde{C}_y$. In order to do that, we merge diagram $10'$ with the analogous diagram 10_y that gives the lifting of C_y, by identifying the corresponding spaces and maps which are the same in the two diagrams (these are $U' = U_y$, $V' = V_y$, $P'/G' = P_y/G_y$, $\widetilde{U}' = \widetilde{U}_y$, $\widetilde{V}' = \widetilde{V}_y$ and the maps between them). We get this way a unique commutative diagram which can be completed, still keeping the commutativity, by the addition of the following maps: the reduction map $P' \to P_y$; a PL map $Q' \to Q_y$ constructed by using the universal property of the connected pullback $q_y : Q_y \to U_y$; a PL map $\widetilde{P}' \to \widetilde{P}_y$, whose existence derives from the minimality of the regularization $r_y : \widetilde{P}_y \to P_y/G_y$. The last map we added clearly gives the desired reduction.

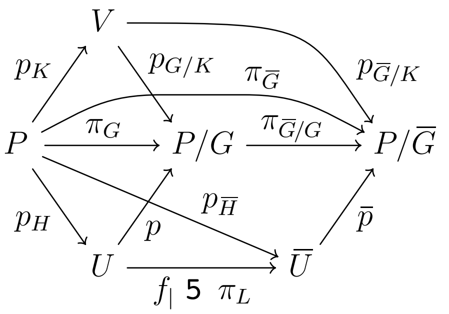
Hence the lifted charts generate a branchfold structure on X turning $f : X \to Y$ into a branchfold covering (f is complete, being a non-degenerate PL map).

The unicity of such branchfold structure immediately follows once we prove that for any branchfold charts C on X and C' on Y, related by diagram 8 with the additional assumption that ψ is an isomorphism, there is a chart equivalence between C and a suitable lifting $\widetilde{C}'$ of C'. Let us look at diagrams 8 and 10′. By Proposition 1.15, U is a connected component of $f^{-1}(U')$. Moreover, up to chart isomorphism, assuming that ψ is an isomorphism is the same as assuming $V = V'$ and $\psi = \mathrm{id}$. Now, let $q' = p_{H'} \circ q'_1 = f_| \circ q'_2 : Q' \to U'$ be the connected pullback of $p_{H'}$ and $f_|$. Then, the universal property of this pullback gives us a branched covering $s'' : P \to Q'$ such that $q_1 \circ s'' = \varphi$ and $q_2 \circ s'' = p_H$. Finally, we put $r'' = \pi_{G'} \circ \varphi$. We get this way a commutative diagram which is like the diagram 10′ realizing the lifting of C' with $\widetilde{U}' = U$. The differences are that we have P, p_H, p_K, s'' and r'' respectively in place of $\widetilde{P}'$, $p_{\widetilde{H}'}$, $p_{\widetilde{K}'}$, s' and r', and that r'' is not necessarily regular. However, by the same argument used in diagram 11, we can construct a new branchfold chart $\widehat{C}$ starting from the minimal regularization $\widehat{r} = r'' \circ \widehat{s} : \widehat{P} \to P/G$ of $r'' : P \to P'/G'$. This new chart dominates C through $\widehat{s} : \widehat{P} \to P$. On the other hand, it dominates also $\widetilde{C}'$ through a map $t : \widehat{P} \to \widetilde{P}'$, which exists by the minimality of r' as a regularization of $\pi_{G'} \circ q'_1$. Thus we have the desired equivalence. □

PROPOSITION 2.17. *Let X be a branchfold and $f : X \to Y$ be a regular branched covering, whose deck transformations are branchfold isomorphisms which preserve local orientations at their fixed points. Then the branchfold structure of X induces a unique branchfold structure on Y turning f into a PL branchfold covering.*

Proof. Given $x \in X$, let L be the group of the deck transformations of f that fix x and let C be any L-invariant conical chart of X centered at x, such that the induced action of L on U is conical and $t(U) \cap U = \emptyset$ for any deck transformation t of f not in L. Since L is finite and acts on X by branchfold isomorphisms, such a chart always exists, an example being the local model of X at x, based on any L-invariant sufficiently small open star centered at x. Then, the restriction $f_| : U \to \overline{U}$ is a conical regular branched covering of an open conical neighborhood $\overline{U} = f(U)$ of $y = f(x)$ in Y, and we can think of it as the canonical projection $\pi_L : U \to U/L \cong \overline{U}$, where the action of L on U is given by restriction and preserves orientations by hypothesis.

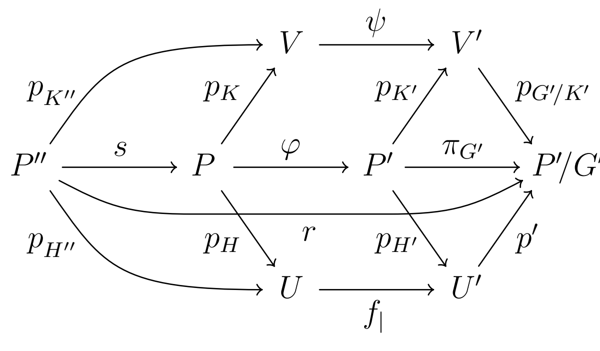
As a consequence of the L-invariance of the chart C, for any $l \in L$ the restriction $l : U \to U$ lifts to some PL homeomorphism $\lambda : P \to P$. Let $\overline{H}$ be the group of all such liftings when l varies in L. Then, $H \trianglelefteq \overline{H}$, since H consists of the liftings of the identity. Moreover, by the L-invariance of U, we see that K, and hence also G, is a normal subgroup in the group $\overline{G}$ of orientation preserving PL homeomorphisms of P generated by $\overline{H} \cup K$. So, we can write $\overline{G} = \overline{H}K$. By Propositions 1.14 and 1.22, all maps in the following commutative diagram are regular branched coverings, possibly except p and $\bar{p}$ (which are however branched coverings). Then $(\overline{U}, P, V, \overline{G} = \overline{H}K)$ is a conical branchfold chart on Y centered at y, which is related to the original chart C by a commutative diagram like diagram 8 (with $\varphi = \mathrm{id}_P$ and $\psi = \mathrm{id}_V$).



(12)

The branchfold chart $(\overline{U}, P, V, \overline{G} = \overline{H}K)$ is well defined up to chart equivalence, being independent on the choices involved in its construction. In particular, if $(U, P', V, G' = H'K')$ is an L-invariant conical chart centered at x which dominates C, then $(\overline{U}, P', V, \overline{G}' = \overline{H}'K')$ dominates $(\overline{U}, P, V, \overline{G} = \overline{H}K)$. Moreover, since the deck transformations of f are branchfold isomorphisms it does not matter what x we choose in $f^{-1}(y)$. The construction also preserves chart restrictions. Namely, if we start from any $x' \in U$ and denote by L' the group of the deck transformations of f that fix x' and by C' any L'-invariant conical restriction of C centered at x', then the resulting chart $(\overline{U}, P', V, \overline{G}' = \overline{H}'K')$ is a restriction of $(\overline{U}, P, V, \overline{G} = \overline{H}K)$. Thus, when y varies in Y the branchfold charts $(\overline{U}, P, V, \overline{G} = \overline{H}K)$ constructed above form a branchfold atlas on Y. This gives us the desired branchfold structure on Y turning f into a branchfold covering (f is complete, being a non-degenerate PL map).

In order to prove the unicity of such branchfold structure, consider any chart C of the branchfold X and any branchfold chart C' on the polyhedron Y, which are related as in diagram 8. Given $x \in U$ and $y = f(x) \in U'$, up to chart restriction we can assume that both charts are conical, respectively centered at x and y, in such a way that $U' \cong U/L$, where L is the group of the deck transformations of f which fix x. Then, let $r : P'' \to P'/G'$ be the minimal regularization of the branched covering $\pi_{G'} \circ \varphi : P \to P'/G'$ and let $p_{H''} : P'' \to U$ and $p_{K''} : P'' \to V$ be defined by composition, in such a way that the following diagram commutes.



(13)

By putting $G'' = H''K''$, where H'' and K'' are the subgroups of the group of deck transformations of r corresponding to $p_{H''}$ and $p_{K''}$ respectively, we get a new chart $(U, P'', V, G'' = H''K'')$ of X, which dominates the original one. On the other hand, this chart is L-invariant and the above construction applied to it gives us a new chart $(\overline{U}, P'', V, \overline{G}'' = \overline{H}''K'')$ on Y, which dominates C' through the map $\varphi \circ s : P'' \to P'$. Hence, we are done thanks to the arbitrary choice of y and C'. $\square$

Propositions 2.16 and 2.17 apply to the branched coverings p_K and p_H of diagram 2 for any local chart C, to determine the branchfold structure of U starting from the smooth one of V. Thanks to Propositions 2.15 and 2.16, we can consider pullbacks and regularizations in the category of PL branchfold coverings, by endowing the new spaces arising in those constructions with the branchfold structure which turns all the new maps into PL branchfold coverings. From now on, we will do that without any further comment.

Universal branchfold covering

Let X be a connected branchfold. We recall that the singular locus ΣX is a good subpolyhedron of X and hence $X - \Sigma X$ is connected by Proposition 1.3.

Given any point $x \in X$, let $(U_x, P, V, G = HK)$ be a conical chart of X centered at x. Referring to diagram 2, we consider the maps $p_| : U_x - p_H(S_G) \to P/G - B_G$ and $p_{G/K|} : V - p_K(S_G) \to P/G - B_G$, where S_G is the singular set of the action of G on P and B_G is the branch set of the canonical projection π_G. These are connected ordinary coverings by Proposition 1.3 and 1.8. On the other hand, $p_H(S_G) \subset U_x$ is a subpolyhedron of codimension ≥ 2 containing the singular set $\Sigma U_x = \Sigma X \cap U_x$. Since $U_x - \Sigma U_x$ is a manifold, by transversality the homomorphism $i_* : \pi_1(U_x - p_H(S_G)) \to \pi_1(U_x - \Sigma U_x)$ induced by the inclusion is surjective. We put $\Gamma_x = i_*((p_{|*})^{-1}(\operatorname{Im} p_{G/K|*})) \trianglelefteq \pi_1(U_x - \Sigma U_x)$. From a different viewpoint, Γ_x is the normal subgroup of $\pi_1(U_x - \Sigma U_x)$ whose elements are represented by the loops ω in $U_x - p_H(S_G)$ with the following property: if $\widetilde{\omega}$ is any path lifting ω to $P - S_G$ through the ordinary covering $p_{H|} : P - S_G \to U_x - p_H(S_G)$, then its projection $p_K \circ \widetilde{\omega}$ in $V - p_K(S_G)$ is a loop. In particular, Γ_x contains $i_*(\operatorname{Im} p_{H|*})$ and hence it has finite index in $\pi_1(U_x - \Sigma U_x)$, being $p_{H|}$ a finite covering. The group Γ_x depends only on the local model of X at x and not on the particular conical chart we started with. In fact, any such chart dominates C_x, giving raise to the same group Γ_x.

Once the base points $* \in X$ and $*_x \in U_x$ for any $x \in X$ are fixed, we choose a path α_x in $X - \Sigma X$ connecting $*$ to $*_x$ and we denote by $h_x : \pi_1(U_x - \Sigma U_x) \to \pi_1(X - \Sigma X)$ the natural homomorphism induced by the map $\omega \mapsto \overline{\alpha}_x \omega \alpha_x$. At this point, we define Γ_X to be the smallest normal subgroup of $\pi_1(X - \Sigma X)$ containing all the $h_x(\Gamma_x)$'s, which is independent on the choice of the α_x's. It is enough to consider the groups $h_x(\Gamma_x)$ with $x \in \Sigma X$, as the other ones are trivial. Actually, due to the fact that the local model is constant on the connected components of the strata of Σ_X, it would suffice to let x vary on a set of representatives of such components.

Definition 2.18. We call $\Gamma_X \trianglelefteq \pi_1(X - \Sigma X)$ the *characteristic group* of X, and $\Gamma_x \trianglelefteq \pi_1(U_x - \Sigma U_x)$ the *local characterictic group* of X at $x \in X$.

Now, we consider the ordinary regular covering $r : R \to X - \Sigma X$ corresponding to the normal subgroup $\Gamma_X \trianglelefteq \pi_1(X - \Sigma X)$, i.e. such that $r_*(\pi_1(R)) = \Gamma_X$. For each $x \in X$ and each connected component A of $r^{-1}(U_x - \Sigma U_x)$, the group $r_{|*}(\pi_1(A)) \leqslant \pi_1(U_x - \Sigma U_x)$ contains the local characteristic group Γ_x. In fact, choose base points $\widetilde{*}$ and $\widetilde{*}_x$ respectively for R and A, such that $r(\widetilde{*}) = *$ and $r(\widetilde{*}_x) = *_x$, and choose

$\alpha_x = r \circ \beta_x$ with β_x a path in R from $\widetilde{*}$ to $\widetilde{*}_x$. Then, by definition of Γ_X, for every loop ω in $U_x - \Sigma U_x$ representing an element of Γ_x, the loop $\overline{\alpha}_x \omega \alpha_x$ lifts to a loop in R with respect to the covering r. This lifting has to be of the form $\overline{\beta}_x \widetilde{\omega} \beta_x$, where $\widetilde{\omega}$ is a loop in A such that $r_| \circ \widetilde{\omega} = \omega$. Hence, since the local characteristic group Γ_x has finite index in $\pi_1(U_x - \Sigma U_x)$, the same is true for the group $r_{|*}(\pi_1(A))$.

We conclude that r satisfies the monodromy hypothesis of Proposition 1.9, therefore it can be completed to a regular branched covering $u_X : \widetilde{X} \to X$. According to Proposition 2.15, we think of $\widetilde{X}$ as a branchfold endowed with the unique structure turning u_X into a PL branchfold covering.

Definition 2.19. We call $u_X : \widetilde{X} \to X$ the *universal branchfold covering* of the connected branchfold X. Moreover, we define the *branchfold fundamental group* as the group $\pi_1^b(X) = \pi_1(X - \Sigma X)/\Gamma_X$ of deck transformations of u_X.

Universal branchfold coverings are PL branchfold coverings and they are natural in the sense specified by the following proposition.

Proposition 2.20. *For any PL branchfold covering $f : X \to Y$, there exists a PL branchfold covering $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ such that $f \circ u_X = u_Y \circ \widetilde{f}$, and this is unique up to deck transformations of u_X and u_Y.*

Proof. Let us consider the good subpoliedra $S = \Sigma Y \cup f(\Sigma X) \subset Y$ and $f^{-1}(S) \subset X$. The restriction $f_| : X - f^{-1}(S) \to Y - S$, as well as the restriction of u_X over $X - f^{-1}(S)$ and that of u_Y over $Y - S$, are all ordinary coverings.

Any loop in $X - f^{-1}(S)$ representing an element of $\operatorname{Im} u_{X|*}$ is a product of the form $\overline{\alpha}_x \omega_x \alpha_x$, where ω_x is a loop in $U_x - f^{-1}(S)$ whose lifting to P_x through p_{H_x} projects to a loop in V_x through p_{K_x}. We put $y = f(x)$, $\alpha_y = f \circ \alpha_x$ and $\omega_y = f \circ \omega_x$. By diagram 8 for the conical restriction $f_| : U_x \to U_y$, we see that ω_y is a loop in $U_y - S$ whose lifting to P_y through p_{H_y} projects to a loop in V_y through p_{K_y}. Hence, $\overline{\alpha}_y \omega_y \alpha_y$ represents an element of $\operatorname{Im} u_{Y|*}$. Therefore, the homomorphism $f_{|*} : \pi_1(X - f^{-1}(S)) \to \pi_1(Y - S)$ sends $\operatorname{Im} u_{X|*}$ into $\operatorname{Im} u_{Y|*}$ and hence the restriction $f_| : X - f^{-1}(S) \to Y - S$ can be lifted to an ordinary covering $\widetilde{f}_| : \widetilde{X} - u_X^{-1}(f^{-1}(S)) \to \widetilde{Y} - u_Y^{-1}(S)$ through the ordinary coverings $u_{X|}$ and $u_{Y|}$. Then, the desired lifting $\widetilde{f}$ is obtained as the completion of $\widetilde{f}_|$. Proposition 2.16, locally applied to $f \circ u_X = u_Y \circ \widetilde{f}$, tells us that $\widetilde{f}$ is actually a branchfold covering. On the other hand, the uniqueness of such a lifting $\widetilde{f}$ up to deck transformations immediately follows from that of $\widetilde{f}_|$. □

Next proposition characterizes universal branchfold coverings in terms of their universal property with respect to a certain class of PL branchfold coverings.

Proposition 2.21. *Let X be a connected branchfold. The universal branchfold covering $u_X : \widetilde{X} \to X$ satisfies the following property: for any $\widetilde{x} \in \widetilde{X}$, the local model C_x of X at $x = u_X(\widetilde{x})$ lifts to a conical (possibly non-reduced) branchfold chart $(U_{\widetilde{x}}, P_x, V_x, \overline{G}_x = \overline{H}_x K_x)$ of $\widetilde{X}$ centered at $\widetilde{x}$, such that $p_{H_x} = u_{X|} \circ p_{\overline{H}_x}$, with $\overline{H}_x \leqslant H_x$ and $\overline{H}_x \cap K_x = H_x \cap K_x$. Moreover, for any PL branchfold covering $f : \widehat{X} \to X$ satisfying the same property, there exists a PL branchfold covering $g : \widetilde{X} \to \widehat{X}$ such that $u_X = f \circ g$.*

Proof. Let x and $\widetilde{x}$ be as in the statement. When defining u_X, we proved that $i_*(\operatorname{Im} p_{H_x|*}) \leqslant \Gamma_x \leqslant \operatorname{Im} u_{X|*} \leqslant \pi_1(U_x - \Sigma U_x)$, where $u_{X|} : U_{\widetilde{x}} - \Sigma U_{\widetilde{x}} \to U_x - \Sigma U_x$ is the restriction of u_X to the non-singular part of the open star $U_{\widetilde{x}}$ of $\widetilde{X}$ at $\widetilde{x}$. By the standard theory of ordinary coverings, there exists a lifting $r_x : P_x - S_{G_x} \to U_{\widetilde{x}} - \Sigma U_{\widetilde{x}}$ of $i \circ p_{H_x|} : P_x - S_{G_x} \to U_x - \Sigma U_x$ through $u_{X|}$. The restriction $r_{x|} : P_x - S_{G_x} \to r_x(P_x - S_{G_x})$ is a regular ordinary covering, which can be completed to a regular branched covering $p_{\overline{H}_x} : P_x \to U_{\widetilde{x}}$, whose deck transformations form a subgroup $\overline{H}_x \leqslant H_x$. By the special case of diagram 10 with $\widetilde{P} \cong Q \cong P = P_x$, $\widetilde{V} = V = V_x$, $\widetilde{U} = U_{\widetilde{x}}$, $U = U_x$, $s \cong q_1 \cong \varphi = \mathrm{id}_{P_x}$, $q_2 \cong p_{\widetilde{H}} = p_{\overline{H}_x}$, $p_{\widetilde{K}} = p_K = p_{K_x}$ and $p_H = p_{H_x}$, we see that $(U_{\widetilde{x}}, P_x, V_x, \overline{G}_x = \overline{H}_x K_x)$ is a conical branchfold chart of $\widetilde{X}$ centered at $\widetilde{x}$. In order to prove the equality $\overline{H}_x \cap K_x = H_x \cap K_x$, let us consider any $g \in H_x \cap K_x$ and let α be any path in $P_x - S_{G_x}$ from the base point $*_x$ to its image $g(*_x)$. Then, $\omega = p_{H_x} \circ \alpha$ is a loop in $U_x - \Sigma U_x$ and $p_{K_x} \circ \alpha$ is a loop in V_x, hence $\omega \in \Gamma_x$. The inclusion $\Gamma_x \leqslant \mathrm{Im}\, u_{X|*}$ implies that $\widetilde{\omega} = p_{\overline{H}_x} \circ \alpha$ is a loop in $U_{\widetilde{x}} - \Sigma U_{\widetilde{x}}$. Hence $g \in \overline{H}_x$.

To prove the second part, let us consider any PL branchfold covering $f : \widehat{X} \to X$ as in the statement. The restriction $f_| : \widehat{X} - f^{-1}(\Sigma X) \to X - \Sigma X$ is an ordinary covering, in such a way that f can be thought as its completion. Let $\bar{\alpha}_x \omega_x \alpha_x$ be any generator of Γ_X, where α_x is a path from the base point $*$ of X to the base point $*_x$ of U_x and ω_x is a loop in $U_x - \Sigma U_x$. For any lifting $\widetilde{\omega}_x$ of ω_x to P_x through p_{H_x}, there exists $g \in H_x \cap K_x$ such that $\widetilde{\omega}_x(1) = g(\widetilde{\omega}_x(0))$. Let $\widehat{\alpha}_x$ be the lifting of α_x through $f_|$ starting from the base point of $\widehat{X} - f^{-1}(\Sigma X)$ and denote by $\widehat{x} \in f^{-1}(x)$ the point such that $\widehat{\alpha}_x(1) \in U_{\widehat{x}}$. The equality $\overline{H}_x \cap K_x = H_x \cap K_x$ for the conical restriction of f at $\widehat{x}$ implies that $\widehat{\omega}_x = p_{\overline{H}_x} \circ \widetilde{\omega}_x$ is a loop in $U_{\widehat{x}}$. Thus, $\bar{\alpha}_x \omega_x \alpha_x$ belongs to $f_{|*}(\pi_1(X - f^{-1}(\Sigma X)))$. So, $\Gamma_X \leqslant \mathrm{Im}\, f_{|*}$. Since Γ_X coincides with $\mathrm{Im}\, u_{X|*}$ for the restriction $u_{X|} : \widetilde{X} - u_X^{-1}(\Sigma X) \to X - \Sigma X$, such restriction lifts through $f_|$ to an ordinary covering $s : \widetilde{X} - u_X^{-1}(\Sigma X) \to \widehat{X} - f^{-1}(\Sigma X)$. Then, the desired factorization $u_X = f \circ g$ can be obtained by defining g as the completion of s over $\widehat{X}$. □

The map φ in diagram 8 can be assumed to be a branchfold isomorphism for any branchfold covering (to see this, consider the dominations of the two involved charts induced by the regularization of the branched covering $\pi_{G'} \circ \varphi$). In other words, we can always assume $P = P'$, $K = K'$ and $H \leqslant H'$. The significant point in the local property stated for u_X by Proposition 2.21 is that in this case the same assumption can be made keeping the local chart of the range to be minimal (no domination is required for it). Moreover, the equality $\overline{H}_x \cap K_x = H_x \cap K_x$ tells us that such a local property is invariant up to dominations/reductions of the local chart of the range.

By Proposition 2.21, when X is a pure branchfold u_X is an ordinary covering, and $\widetilde{X}$ is a pure branchfold as well. In general $\widetilde{X}$ is not a pure branchfold, but we can say that it is as pure as possible among the simply connected PL branchfold coverings of X whose branch set is contained in ΣX. This fact can be formalized as a universal property of u_X with respect to the pure branchfold coverings of X.

Proposition 2.22. *Let X be a connected branchfold. Then the universal covering space $\widetilde{X}$ is simply connected. Moreover, for any PL branchfold covering $f : \widehat{X} \to X$ with $\widehat{X}$ a simply connected pure branchfold there exists a PL branchfold covering $g : \widehat{X} \to \widetilde{X}$ such that $f = u_X \circ g$.*

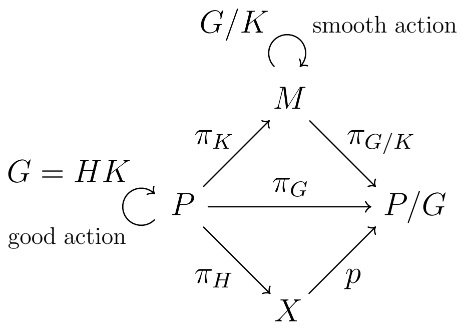
Proof. The simply connectedness of $\widetilde{X}$ follows by contradiction from Proposition 2.21. Now, by applying Proposition 2.20 to the PL branched covering $f : \widehat{X} \to X$ in the statement, we get the factorizing map g as the lifting $\widetilde{f}$ of f to the universal branchfold coverings (up to branchfold isomorphism), being the universal branchfold covering of $\widehat{X}$ a branchfold isomorphism by the above observation. □

Another consequence of Proposition 2.21 is that, when X is a connected orbifold, $u_X : \widetilde{X} \to X$ coincides with the universal orbifold covering and $\pi_1^b(X)$ coincides with the fundamental orbifold group $\pi_1^o(X)$. Furthermore, in this case the characteristic group Γ_X is normally generated by the powers $\mu_A^{i_A}$, where A varies among the connected components of the $(m-2)$-stratum of ΣX, μ_A is any meridian around A and i_A is the index of A. In fact, in this case $\Gamma_x = \operatorname{Im} p_{H_x|*}$ and the total space P_x of the local model at x is a disk for any $x \in X$. Then, we can easily conclude that any loop $\bar{\alpha}_x \omega \alpha_x \in h_x(\Gamma_x)$ is homotopic to a product of powers $\mu_A^{i_A}$ as above, being any loop $\widetilde{\omega}$ in $P_x - p_{H_x}^{-1}(\Sigma U_x)$ homotopic to the composition of meridians around $p_{H_x}^{-1}(\Sigma U_x)$.

Good branchfolds

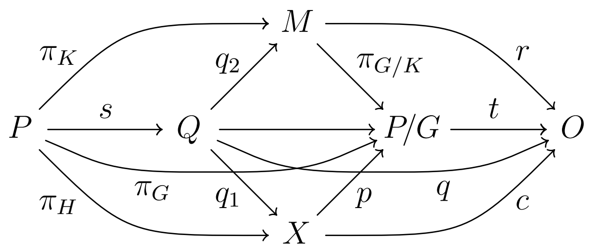
Thanks to Propositions 2.16 and 2.17, we can extend to branchfolds the construction of locally orientable good orbifolds as global quotients of smooth manifolds by properly discontinuous smooth actions.

Namely, let P be a connected polyhedron and $G = HK$ be a group acting on P, such that $K \trianglelefteq G$ is normal, $M = P/K$ is a smooth m-manifold and the induced action of G/K on M is smooth and preserves local orientations, i.e. it is given by diffeomorphisms which preserve local orientations at their fixed points. By Proposition 2.16 we endow P with a pure m-branchfold structure, which is uniquely determined by the property of turning the canonical projection $\pi_K : P \to M$ into a PL branchfold covering. The action of H on P leaves this branchfold structure invariant and preserves local orientations. Hence by Proposition 2.17 we get a unique m-branchfold structure on $X = P/H$, which turns the canonical projection $\pi_H : P \to X$ into a PL branchfold covering. On the other hand, the quotient space $P/G = M/(G/K)$ is a locally orientable good m-orbifold and $p : X \to P/G$ is a PL branchfold covering. Hence we have diagram 14, made of PL branchfold coverings.



(14)

Actually, starting from any PL branchfold covering $c : X \to O$ of a good orbifold O, we can produce a similar diagram in the following way (see diagram 15). Let $r : M \to O$ be a regular orbifold covering with M a smooth manifold, which always exists being any manifold covering of O virtually regular (cf. [11]). Then, consider the connected pullback $q : Q \to O$ of r and c, and the minimal regularization $q \circ s : P \to O$ of q. Denote by L the group of deck transformations of such regularization, and define $H \leqslant L$ as the subgroup corresponding to the regular covering $\pi_H = q_1 \circ s$, and $K \trianglelefteq L$ as the normal subgroup corresponding to the regular covering $\pi_K = q_2 \circ s$. Finally, put $G = HK \leqslant L$ and complete the diagram with the quotient P/G and the coverings p, t and $\pi_{G/K}$. The new spaces Q, P and P/G are endowed with the branchfold structures which turn all the maps into PL branchfold coverings. In particular, $P/G = M/(G/K)$ turns out to be an orbifold. If $c : X \to O$ is a finite



(15)

branchfold covering of a very good orbifold, then all the coverings in diagram 15 can be assumed to be finite. Hence we get a diagram like 14, made of finite coverings.

In the light of the considerations above, the notions introduced by the next definition are natural extensions of the usual notions of good and very good orbifold.

Definition 2.23. A connected branchfold X is called a *good branchfold* if it admits PL branchfold covering $c : X \to O$ onto a good orbifold O. If in addition c has finite degree and O is a very good orbifold, then we call X a *very good branchfold*.

These notions coincide with the usual ones when referred to orbifolds. In fact, it is clear from the definition that a good (resp. very good) orbifold X is also good (resp. very good) as a branchfold. For the viceversa, we prove that when X is an orbifold the branchfold P in diagram 15 is a smooth manifold (i.e. $\Sigma P = \emptyset$). To see this, we consider the universal (orbifold) covering $u_O : \widetilde{O} \to O$. If X is an orbifold, both p and c are orbifold coverings, hence u_O lifts to orbifold coverings $\widetilde{O} \to M$ and $\widetilde{O} \to X$. By the universal properties of the pullback and of the minimal regularization, we get a branchfold covering $\widetilde{O} \to P$. Hence P is a manifold, being simoultaneously a pure branchfold (as a covering of M) and an orbifold (as a quotient of $\widetilde{O}$).

The universal branchfold covering $\widetilde{X}$ of a good branchfold X is a pure branchfold, since the covering $c : X \to O$ in Definition 2.23 lifts to a branchfold covering $\widetilde{c} : \widetilde{X} \to \widetilde{O}$ with $\widetilde{O}$ a smooth manifold. Moreover, by the definition of branchfold chart, any branchfold is locally very good, being diagram 2 at page 10 a special case of diagram 14, with U in place of X and all the coverings of finite degree.

3. Geometric branchfolds

In this section we introduce the notion of geometric structure on a branchfold and extend to geometric branchfolds the well known goodness theorem for geometric orbifolds ([18], [12]). Then, we relate conifolds and branchfolds endowed with geometries modelled on constant curvature spaces. In particular, we apply the geometric goodness theorem to establish which conifolds can be thought as branchfolds.

By a *geometry* we mean a pair $(\mathcal{G}, \mathcal{M})$, where $\mathcal{M}$ is a simply connected smooth manifold and $\mathcal{G}$ is a transitive group of diffeomorphisms of $\mathcal{M}$ which is *locally effective*, meaning that if $g \in \mathcal{G}$ and $g_{|U} = \mathrm{id}_U$ for some open subset $U \subset \mathcal{M}$ then $g = \mathrm{id}_{\mathcal{M}}$. Following [12], here we use local effectiveness in place of the stronger assumption of analyticity of the action made by Thurston in his notes [17].

Given a $(\mathcal{G}, \mathcal{M})$, we denote by $\widehat{\mathcal{G}}$ the pseudo-group of the diffeomorphisms between non-empty open subsets of $\mathcal{M}$ obtained as restrictions of elements of $\mathcal{G}$, i.e. $\widehat{\mathcal{G}} =$

$\{g_| : U \to g(U) \mid g \in \mathcal{G} \text{ and } U \neq \emptyset \text{ open in } \mathcal{M}\}$. The local effectiveness can be reformulated by saying that each element of $\widehat{\mathcal{G}}$ extends to an unique elements of $\mathcal{G}$.

A polyhedron $P \subset \mathcal{M}$ is called *$\mathcal{G}$-admissible* if it admits a stratification, whose strata locally coincide with fixed point sets $\operatorname{Fix} G$, where $G \subset \widehat{\mathcal{G}}$ is any finitely generated group acting on an open subset of $\mathcal{M}$. In particular, the singular set S_G of any good action of a finite group $G \subset \widehat{\mathcal{G}}$ on an open subset $V \subset \mathcal{M}$ is $\mathcal{G}$-admissible in $\mathcal{M}$. In fact, for every $x \in S_G$, the fixed point set $\operatorname{Fix} G_x$ of the stabilizer of x is a smooth submanifold of V contained in S_G, and we can define a stratification of S_G by putting $(S_G)_i = \{x \in S_G \mid \dim \operatorname{Fix} G_x \leq i\}$. Such a stratification satisfies the property required for the $\mathcal{G}$-admissibility of S_G. In the case of a Riemannian geometry, i.e. when $\mathcal{M}$ is a Riemannian manifold and $\mathcal{G}$ is the group of isometries of $\mathcal{M}$, if $P \subset \mathcal{M}$ is $\mathcal{G}$-admissible then it can be stratified by totally geodesic submanifolds of $\mathcal{M}$ (cf. [10]). The viceversa holds when $\mathcal{M}$ has constant curvature. Our definition is a tentative reformulation of this metric property in the context of $(\mathcal{G}, \mathcal{M})$ geometries.

Definition 3.1. A branchfold chart C is called a *$(\mathcal{G}, \mathcal{M})$-chart* when V is identified with an open subset $V \subset \mathcal{M}$ in such a way that:
(1) the branch set of p_K is a $\mathcal{G}$-admissible subpolyhedron of $\mathcal{M}$;
(2) the induced action of G/K on V is given by elements of $\widehat{\mathcal{G}}$.

Property 1 is aimed to impose a reasonable restriction on the branched covering π_K, otherwise any branched covering of a connected open subset $V \subset \mathcal{M}$ would be a $(\mathcal{G}, \mathcal{M})$-chart for any geometry $(\mathcal{G}, \mathcal{M})$. As we will see, this property works well in the case of constant curvature Riemannian geometries, but we are not sure that it is the right property to be required in general. On the other hand, property 2 is a quite natural extension of the analogous one usually required for an orbifold chart to be geometric. Actually, it implies that the definition of $(\mathcal{G}, \mathcal{M})$-chart reduces to the standard one in the orbifold case.

Two $(\mathcal{G}, \mathcal{M})$-charts are called *(strongly) $(\mathcal{G}, \mathcal{M})$-isomorphic* when they are (strongly) isomorphic and the diffeomorphism $V_1 \cong V_2$ belongs to $\widehat{\mathcal{G}}$.

Any restriction of a $(\mathcal{G}, \mathcal{M})$-chart is still a $(\mathcal{G}, \mathcal{M})$-chart. In fact, referring to Definition 2.2, $B_{p_{K'}} = B_{p_K} \cap V'$ and this is $\mathcal{G}$-admissible, since $\mathcal{G}$-admissibility is a local property. The action of G'/K' on V' is given by elements of $\widehat{\mathcal{G}}$, being a restriction of the action of G/K on V. Analogously, any reduction of a $(\mathcal{G}, \mathcal{M})$-chart is still a $(\mathcal{G}, \mathcal{M})$-chart. In this case, referring to Definition 2.3, B_{p_K} is a subpolyhedron of $B_{p_{K'}}$ and it can be obtained by deleting some connected components from the strata of the stratification giving the admissibility of $B_{p_{K'}}$, hence it is admissible itself. At the same time, property 2 is trivially preserved by reductions.

On the contrary, it is not difficult to see that a chart dominating a $(\mathcal{G}, \mathcal{M})$-chart is not necessarily a $(\mathcal{G}, \mathcal{M})$-chart. In particular, the common dominating chart giving the equivalence between two $(\mathcal{G}, \mathcal{M})$-charts is not necessarily a $(\mathcal{G}, \mathcal{M})$-chart.

Definition 3.2. A branchfold atlas $\mathcal{A} = \{C_i\}_{i \in I}$ on X is called a *$(\mathcal{G}, \mathcal{M})$-atlas*, if it consists of $(\mathcal{G}, \mathcal{M})$-charts which satisfy the compatibility condition in Definition 2.5, where the strong isomorphisms are required to be strong $(\mathcal{G}, \mathcal{M})$-isomorphism (i.e. the diffeomorphisms $V_i' \cong V' \cong V_j'$ in diagram 7 belong to $\widehat{\mathcal{G}}$). A maximal $(\mathcal{G}, \mathcal{M})$-atlas on X is called *$(\mathcal{G}, \mathcal{M})$-structure*.

Definition 3.3. By a *geometric branchfold* modelled on the geometry $(\mathcal{G}, \mathcal{M})$ (or a $(\mathcal{G}, \mathcal{M})$-*branchfold*) we mean a pair $X_{\mathcal{S}} = (X, \mathcal{S})$, where $X = X_{\mathcal{B}}$ is a branchfold and $\mathcal{S} \subset \mathcal{B}$ is a $(\mathcal{G}, \mathcal{M})$-structure on X. We will write X instead of $X_{\mathcal{S}}$, if no confusion can arise. Moreover, when talking of a chart (resp. an atlas) of a $(\mathcal{G}, \mathcal{M})$-branchfold $X = X_{\mathcal{S}}$, we will always assume that it is a $(\mathcal{G}, \mathcal{M})$-chart (resp. a $(\mathcal{G}, \mathcal{M})$-atlas) in $\mathcal{S}$.

Any $(\mathcal{G}, \mathcal{M})$-atlas uniquely estends to a $(\mathcal{G}, \mathcal{M})$-structure (by the argument used at page 12). This is essentially due to the fact that restricion preserves $(\mathcal{G}, \mathcal{M})$-charts.

Definition 3.4. A branchfold map $f : X \to Y$ between $(\mathcal{G}, \mathcal{M})$-branchfolds is called a $(\mathcal{G}, \mathcal{M})$-*map* if for every $(\mathcal{G}, \mathcal{M})$-charts C of X and C' of Y the map $\psi : V \to V'$ in diagram 8 belongs to $\widehat{\mathcal{G}}$. By $(\mathcal{G}, \mathcal{M})$-*covering* (resp. $(\mathcal{G}, \mathcal{M})$-*isomorphism*) we mean a $(\mathcal{G}, \mathcal{M})$-map which is a branchfold covering (resp. isomorphism).

The notions of $(\mathcal{G}, \mathcal{M})$-branchfold and $(\mathcal{G}, \mathcal{M})$-map (resp. covering) defined above restrict to the usual ones when referred to orbifolds/manifolds and maps (resp. coverings) between them.

For any point $x \in X$ of a $(\mathcal{G}, \mathcal{M})$-branchfold X, there exist arbitrarily small reduced conical $(\mathcal{G}, \mathcal{M})$-charts. We call any of these reduced conical charts a *local* $(\mathcal{G}, \mathcal{M})$-*model* of X at x. This is uniquely determined up to $(\mathcal{G}, \mathcal{M})$-isomorphisms and conical restrictions (which are not necessarily $(\mathcal{G}, \mathcal{M})$-isomorphisms).

Proposition 2.15 doesn't hold in the geometric context. Namely, given a branched covering $f : X \to Y$ onto a $(\mathcal{G}, \mathcal{M})$-branchfold Y, it does not necessarily exist a $(\mathcal{G}, \mathcal{M})$-structure of X turning f into a $(\mathcal{G}, \mathcal{M})$-covering. In fact, when lifting a $(\mathcal{G}, \mathcal{M})$-chart of Y through f, we cannot always guarantee that property 1 of Definition 3.1 is preserved. However, the following proposition says that the universal branchfold covering of a $(\mathcal{G}, \mathcal{M})$-branchfold can be thought as a $(\mathcal{G}, \mathcal{M})$-covering.

Proposition 3.5. *Let X be a connected $(\mathcal{G}, \mathcal{M})$-branchfold. Then the $(\mathcal{G}, \mathcal{M})$-structure of X can be lifted to a unique $(\mathcal{G}, \mathcal{M})$-structure on $\widetilde{X}$ turning the universal branchfold covering $u_X : \widetilde{X} \to X$ into a $(\mathcal{G}, \mathcal{M})$-covering.*

Proof. By Proposition 2.21, any local $(\mathcal{G}, \mathcal{M})$-model C_x of X at $x = u_X(\widetilde{x})$ lifts to a conical $(\mathcal{G}, \mathcal{M})$-chart $(U_{\widetilde{x}}, P_x, V_x, \overline{G}_x = \overline{H}_x K_x)$ on $\widetilde{X}$. In fact, the branched covering p_{K_x} is the same in both the charts, while the action of $\overline{G}_x / K_x$ on V_x is a restriction of that of G_x / K_x. Now, we can argue as in the proof of Proposition 2.16, with the extra requirement that all the chart isomorphisms are $(\mathcal{G}, \mathcal{M})$-isomorphisms, to show that these lifted conical $(\mathcal{G}, \mathcal{M})$-charts form a $(\mathcal{G}, \mathcal{M})$-atlas on $\widetilde{X}$ and that the induced $(\mathcal{G}, \mathcal{M})$-structure is the only one which turns u_X into a $(\mathcal{G}, \mathcal{M})$-covering. □

Next proposition is the geometric version of Proposition 2.17, and can be proven by the very same argument.

Proposition 3.6. *Let X be a $(\mathcal{G}, \mathcal{M})$-branchfold and $f : X \to Y$ be a regular branched covering, whose deck transformations are $(\mathcal{G}, \mathcal{M})$-isomorphisms which preserve local orientations at their fixed points. Then the $(\mathcal{G}, \mathcal{M})$-structure of X induces a unique $(\mathcal{G}, \mathcal{M})$-structure on Y turning f into a $(\mathcal{G}, \mathcal{M})$-covering.*

We now briefly recall some notions concerning $(\mathcal{G}, \mathcal{M})$-manifolds, such as those of holonomy and developing map. We refer to [18] and [12] for more details.

Let M be a connected $(\mathcal{G}, \mathcal{M})$-manifold. A reduced $(\mathcal{G}, \mathcal{M})$-chart C of M can be thought as $(U, \varphi = p_K \circ p_H^{-1} : U \to V)$, being G the trivial group. We will denote by (U, φ) such a chart. Given any path $\alpha : [0,1] \to M$, we consider a sequence $0 = t_0 \leq t_1 \leq \ldots \leq t_k = 1$ such that $\alpha([t_{i-1}, t_i])$ is contained in some reduced $(\mathcal{G}, \mathcal{M})$-chart (U_i, φ_i) of M, for every $i = 1, \ldots, k$. Taking into account the compatibility condition between $(\mathcal{G}, \mathcal{M})$-charts, we can assume that φ_i and φ_{i+1} coincide in a neighborhood of $\alpha(t_i)$ (up to composition by elements of $\mathcal{G}$). The local effectiveness of the action of $\mathcal{G}$ on $\mathcal{M}$ implies that this can be done in a unique way, once the first chart (U_1, φ_1) is choosen. Then, we can define a path $\alpha_{\mathcal{M}} : [0,1] \to \mathcal{M}$, by putting $\alpha_{\mathcal{M}}(t) = \varphi_i(\alpha(t))$ for $t \in [t_{i-1}, t_i]$. Moreover, we can define a continuous family of local $(\mathcal{G}, \mathcal{M})$-models $\{(U'_t, \varphi'_t)\}_{t \in [0,1]}$, such that (U'_t, φ'_t) is the local model at $\alpha(t)$ induced by (U_i, φ_i) if $t \in [t_{i-1}, t_i]$, in such a way that $\alpha_{\mathcal{M}}(t) = \varphi'_t(\alpha(t))$ for every $t \in [0,1]$. By the local effectiveness of the action of $\mathcal{G}$ on $\mathcal{M}$, the path $\alpha_{\mathcal{M}}$ and the family $\{(U'_t, \varphi'_t)\}_{t \in [0,1]}$ are well defined up to multiplication by elements of $\mathcal{G}$, depending only on the choice of the local model (U'_0, φ'_0) at the starting point $\alpha(0)$.

The *holonomy* $H_M : \pi_1(M) \to \mathcal{G}$ of the $(\mathcal{G}, \mathcal{M})$-manifold M is the homomorphism defined as follows, once a local model (U_*, φ_*) at the base point $*$ is fixed. For any loop ω in $(M, *)$ we consider the family of local models $\{(U'_t, \varphi'_t)\}_{t \in [0,1]}$ constructed as above, starting from $(U'_0, \varphi'_0) = (U_*, \varphi_*)$. By the compatibility condition among $(\mathcal{G}, \mathcal{M})$-charts, there exists a unique $g \in \mathcal{G}$ such that $\varphi'_1 = g \circ \varphi'_0$ in a neighborhood of $*$. Then, we put $H_M([\omega]) = g$. The holonomy H_M is defined only up to conjugation in $\mathcal{G}$, depending on the choice of the local model (U_*, φ_*) at the base point.

When the holonomy is trivial, we can define a $(\mathcal{G}, \mathcal{M})$-map $D_M : M \to \mathcal{M}$, by putting $D_M(x) = \alpha_{\mathcal{M}}(1)$, where $\alpha_{\mathcal{M}}$ is the path in $\mathcal{M}$ associated by the above contruction, with $\varphi'_0 = \varphi_*$, to any path α in M from $*$ to x. This is called a *developing map* for M. Different choices of (U_*, φ_*) lead to developing maps which differ by an element of $\mathcal{G}$. Namely, by the local effectiveness of the action of $\mathcal{G}$ on $\mathcal{M}$, for any other $(\mathcal{G}, \mathcal{M})$-map $D'_M : M \to \mathcal{M}$ there exists an element $g \in \mathcal{G}$ such that $D'_M = g \circ D_M$. In particular, for any connected $(\mathcal{G}, \mathcal{M})$-manifold M there always exists the developing map $D_{\widetilde{M}} : \widetilde{M} \to \mathcal{M}$ defined on the universal covering $\widetilde{M}$ of M. Then, for every path $\alpha : [0,1] \to M$, we can realize $\alpha_{\mathcal{M}}$, up to multiplication by elements of $\mathcal{G}$, as $D_{\widetilde{M}} \circ \widetilde{\alpha}$, where $\widetilde{\alpha} : [0,1] \to \widetilde{M}$ is a lifting of α through $u_M : \widetilde{M} \to M$.

Given a connected $(\mathcal{G}, \mathcal{M})$-branchfold X, we consider $X - \Sigma X$ as a connected $(\mathcal{G}, \mathcal{M})$-manifold with the $(\mathcal{G}, \mathcal{M})$-structure induced by the inclusion in X.

Definition 3.7. We denote by $H_X : \pi_1(X - \Sigma X) \to \mathcal{G}$ the holonomy of $X - \Sigma X$ and we call it the *holonomy* of the $(\mathcal{G}, \mathcal{M})$-branchfold X. Moreover, we denote by $H_x : \pi_1(U_x - \Sigma U_x) \to \mathcal{G}$ the holonomy of the conical neighborhood U_x of $x \in X$ and we call it the *local holonomy* of X at x.

Finally, we give a notion of completeness for $(\mathcal{G}, \mathcal{M})$-branchfolds. The above construction of the path $\alpha_{\mathcal{M}}$ can be adapted, with a possibly infinite sequence of t_i's, to associate to any half open path $\alpha : [0,1[\to M$ an open path $\alpha_{\mathcal{M}} : [0,1[\to \mathcal{M}$ well defined up to multiplication by elements of $\mathcal{G}$.

Definition 3.8. A $(\mathcal{G}, \mathcal{M})$-branchfold X is called *complete* when any half open path $\alpha : [0,1[\to X - \Sigma X$, such that $\alpha_{\mathcal{M}} : [0,1[\to \mathcal{M}$ completes to a rectifiable path $\overline{\alpha}_{\mathcal{M}} : [0,1] \to \mathcal{M}$, admits a (rectifiable) completion $\overline{\alpha} : [0,1] \to X$.

In the definition we can equivalently adopt the notion of rectifiability in $\mathcal{M}$ as a smooth manifold or as a polyhedron, with the polyhedral structure given by any smooth triangulation of it. Furthermore, when $\mathcal{M}$ admits a $\mathcal{G}$-invariant Riemannian metric, the $(\mathcal{G},\mathcal{M})$-manifolds $X-\Sigma X$ and $R\subset\widetilde{X}$ can be endowed (in a unique way) with Riemannian metrics which make $D_R : R\to\mathcal{M}$ and $u_{|X} : R\to X-\Sigma X$ into local isometries. The corresponding geodesic distances can be completed by continuity to distances on X and $\widetilde{X}$. So, it makes sense to compare our notion of completeness with the metric one, and a standard argument shows that they coincide (cf. [18]). In particular, if $\mathcal{M}$ admits a $\mathcal{G}$-invariant Riemannian metric, then any compact $(\mathcal{G},\mathcal{M})$-branchfold is complete. By the following proposition, this is true even if such a $\mathcal{G}$-invariant metric does not exist.

Proposition 3.9. *Any compact* $(\mathcal{G},\mathcal{M})$*-branchfold is complete.*

Proof. Let $\alpha : [0,1[\,\to X-\Sigma X$ be such that $\alpha_{\mathcal{M}} : [0,1[\,\to\mathcal{M}$ completes to a rectifiable path $\bar\alpha_{\mathcal{M}} : [0,1]\to\mathcal{M}$. Then, $\alpha_{\mathcal{M}} = D_R\circ\widetilde{\alpha}$, where $\widetilde{\alpha} : [0,1[\,\to R$ is a lifting of α through the ordinary covering $u_{X|} : R\to X-\Sigma X$. Since $\bar\alpha_{\mathcal{M}}$ is rectifiable, $\alpha_{\mathcal{M}}$ has finite length with respect to any Riemannian metric on $\mathcal{M}$. This metric can be lifted to R through the local diffeomorphism D_R, and $\widetilde{\alpha}$ has finite length with respect to this lifted metric, having the same length of $\alpha_{\mathcal{M}}$. On the other hand, $\widetilde{\alpha}([0,1[)$ is contained in a compact subpolyhedron $C\subset\widetilde{X}$, so it is rectifiable with respect to the polyhedral structure of $\widetilde{X}$. Then, α is rectifiable with respect to the polyhedral structure of X, being u_X a PL map. By the compactness of X, α admits some limit point $x\in X$, i.e. there exists a sequence $t_n\to 1$ such that $\lim_{n\to\infty}\alpha(t_n) = x$. Such limit point must be unique, otherwise α would not be rectifiable. □

The geometric goodness theorem

The universal branchfold covering $u_X : \widetilde{X}\to X$ of a connected $(\mathcal{G},\mathcal{M})$-branchfold X can be thought as a $(\mathcal{G},\mathcal{M})$-covering, by putting on $\widetilde{X}$ the $(\mathcal{G},\mathcal{M})$-branchfold structure given by Proposition 3.5. The restriction of u_X over $X-\Sigma X$ is the ordinary regular covering $r : R\to X-\Sigma X$ corresponding to the characteristic group $\Gamma_X \trianglelefteq \pi_1(X-\Sigma X)$, which we introduced in order to define u_X as its completion. Then, $R = \widetilde{X} - u_X^{-1}(\Sigma X)$ can be endowed with the $(\mathcal{G},\mathcal{M})$-manifold structure induced by the inclusion in $\widetilde{X}$. This turns r into a $(\mathcal{G},\mathcal{M})$-covering between $(\mathcal{G},\mathcal{M})$-manifolds.

Proposition 3.10. *If X is a connected* $(\mathcal{G},\mathcal{M})$*-branchfold, then* $\Gamma_x = \operatorname{Ker} H_x$ *for every $x\in X$ and* $\Gamma_X \leqslant \operatorname{Ker} H_X$*. Hence, the holonomy H_R of the* $(\mathcal{G},\mathcal{M})$*-manifold* $R = \widetilde{X} - u_X^{-1}(\Sigma X)$ *is trivial and R admits a developing map* $D_R : R\to\mathcal{M}$.

Proof. Given $x\in X$ and any conical chart $(U_x, P, V, G = HK)$ centered at x, $\Gamma_x = i_*((p_{|*})^{-1}(\operatorname{Im} p_{G/K|*}))$ by definition, where $p_| : U_x - p_H(S_g)\to P/G - B_G$ and $p_{G/K|} : V - p_K(S_G)\to P/G - B_G$ are the ordinary coverings of $(\mathcal{G},\mathcal{M})$-manifolds given by restrictions of the branched coverings p and $p_{G/K}$ of the chart, and $i : U_x - p_H(S_G)\to U_x - \Sigma U_x$ is the inclusion. The holonomy $H_{V-p_K(S_G)}$ is trivial, being $V - p_K(S_G)$ an open subset of $\mathcal{M}$. Then, the equality $H_{V-p_K(S_G)} = H_{P/G-B_G}\circ p_{G/K|*}$ implies that $\operatorname{Im} p_{G/K|*} \leqslant \operatorname{Ker} H_{P/G-B_G}$. On the other hand, for any loop ω in $P/G - B_G$, we can construct $\omega_{\mathcal{M}}$ inside $V - p_K(S_G)\subset\mathcal{M}$ as a lifting of ω through the ordinary covering $p_{G/K|}$. In particular, if $[\omega]\in\operatorname{Ker} H_{P/G-B_G}$ then $\omega_{\mathcal{M}}$ must be a loop in $V - p_K(S_G)$, hence $[\omega]\in\operatorname{Im} p_{G/K|*}$. This proves that $\operatorname{Im} p_{G/K|*} = \operatorname{Ker} H_{P/G-B_G}$. Therefore, $\Gamma_x = i_*((p_{|*})^{-1}(\operatorname{Im} p_{G/K|*})) = i_*((p_{|*})^{-1}(\operatorname{Ker} H_{P/G-B_G}) = i_*(\operatorname{Ker} H_{U_x-p_H(S_G)}) = \operatorname{Ker} H_x$, where the second equality derives from $H_{U_x-p_H(S_G)} =$

$H_{P/G-B_G} \circ p_{|*}$ and the last one from the surjectivity of i_*. Hence, since Γ_X is normally generated by the groups $h_x(\Gamma_x)$ with $x \in X$, the inclusion $\Gamma_X \leqslant \operatorname{Ker} H_X$ immediately follows from the fact that $h_x(\operatorname{Ker} H_x)$ is obviously contained in $\operatorname{Ker} H_X$ for every $x \in X$. We can conclude that H_R is trivial, since $H_R = H_X \circ r_*$ and $\operatorname{Im} r_* = \Gamma_X$, where r is the restriction of u_X over $X - \Sigma X$ (cf. discussion above). □

As a consequence, H_X factorizes through a *holonomy representation* $R_X : \pi_1^b(X) \to \mathcal{G}$ of the branchfold fundamental group $\pi_1^b(X) = \pi_1(X - \Sigma X)/\Gamma_X$, such that $D_R \circ \gamma = R_X(\gamma) \circ D_R$ for every $\gamma \in \pi_1^b(X)$. In other words, D_R is R_X-equivariant.

PROPOSITION 3.11. *Let X be a connected $(\mathcal{G},\mathcal{M})$-branchfold. Then the developing map $D_R : R \to \mathcal{M}$ extends to an R_X-equivariant $(\mathcal{G},\mathcal{M})$-map $C_X : \widetilde{X} \to \mathcal{M}$. Moreover, C_X is a $(\mathcal{G},\mathcal{M})$-covering if and only if X is complete.*

Proof. We define the map C_X by local completion of R_X. Namely, given any $\widetilde{x} \in \widetilde{X} - R$, we consider the local model C_x of X at $x = u_X(\widetilde{x})$ and the conical chart $(U_{\widetilde{x}}, P_x, V_x, \overline{G}_x = \overline{H}_x K_x)$ of $\widetilde{X}$ centered at $\widetilde{x}$, as in Proposition 2.21. Then, $D_R \circ p_{\overline{H}_x}$ and p_{K_x} both restrict to a developing map of the $(\mathcal{G},\mathcal{M})$-manifold $P_x - S_{G_x}$. Hence, there exists $g \in \mathcal{G}$ such that $D_R \circ p_{\overline{H}_x|} = g \circ p_{K_x|} : P_x - S_{G_x} \to V_x - p_{K_x}(S_{G_x})$. Therefore $\overline{H}_x \leqslant K_x$ and we can identify $P_x/\overline{G}_x$ with $V_x \subset \mathcal{M}$. Under this identification, the completion of $D_{R|} : U_{\widetilde{x}} - p_{\overline{H}_x}(S_{G_x}) \to \mathcal{M}$ is given by $g \circ \bar{p}_x : U_{\widetilde{x}} \to \mathcal{M}$, where $\bar{p}_x : U_{\widetilde{x}} \to P_x/\overline{G}_x$ is the covering associated to the chart $(U_{\widetilde{x}}, P_x, V_x, \overline{G}_x = \overline{H}_x K_x)$. The unicity of completions guarantees that all this local completions fit together to give a locally PL map $C_X : \widetilde{X} \to \mathcal{M}$ that extends R_X. Actually, the construction above also tells us that C_X is a $(\mathcal{G},\mathcal{M})$-map.

Now, assume that X is complete. In order to conclude that C_X is a $(\mathcal{G},\mathcal{M})$-covering, we have to prove its completeness with respect to lifting of rectifiable paths. Let $\alpha : [0,1] \to \mathcal{M}$ be a rectifiable path and $\widetilde{\alpha}_| : [0,1[\to \widetilde{X}$ be a partial lifting of it through C_X. Since $u_X^{-1}(\Sigma X)$ is a good subcomplex of $\widetilde{X}$, we can perturb $\widetilde{\alpha}_|$ to a half open path $\beta : [0,1[\to R = \widetilde{X} - u_X^{-1}(\Sigma X)$ such that $D_R \circ \beta$ is rectifiable and $\lim_{t\to 1} d(\beta(t), \widetilde{\alpha}_|(t))) = 0$ for some metric d on $\widetilde{X}$. Then, for $u_X \circ \beta : [0,1[\to X - \Sigma X$, $(u_X \circ \beta)_{\mathcal{M}} = D_R \circ \beta$ is rectifiable. Hence, by the completeness of X, there exists an extension $\gamma : [0,1] \to X$ of $u_X \circ \beta$. Since u_X is a branched covering, γ lifts to $\widetilde{\gamma} : [0,1] \to \widetilde{X}$, which extends β. Then, there exists $\lim_{t\to 1} \widetilde{\alpha}_|(t) = \lim_{t\to 1} \beta(t) = \widetilde{\gamma}(1) \in \widetilde{X}$ and we can complete $\widetilde{\alpha}_|$ to a lifting $\widetilde{\alpha}$ of α. Viceversa, assume that C_X is a $(\mathcal{G},\mathcal{M})$-covering. Let $\alpha : [0,1[\to X - \Sigma X$ be such that $\alpha_{\mathcal{M}}$ completes to a rectifiable path $\bar{\alpha}_{\mathcal{M}} : [0,1] \to \mathcal{M}$. Let $\widetilde{\alpha} : [0,1[\to R$ a lifting of α through the ordinary covering $r = u_{X|} : R \to X - \Sigma X$. Then, $C_X \circ \widetilde{\alpha} = g \circ \alpha_{\mathcal{M}}$, for a suitable $g \in \mathcal{G}$, is rectifiable. By the completeness of C_X, $\widetilde{\alpha}$ admits a completion and so also $\alpha = u_X \circ \widetilde{\alpha}$ does. □

Now we are ready for the announced geometric goodness theorem.

THEOREM 3.12. *Any connected compact $(\mathcal{G},\mathcal{M})$-branchfold X, whose holonomy group $\mathcal{H}_X = R_X(\pi_1^b(X)) = H_X(\pi_1(X - \Sigma X)) \leqslant \mathcal{G}$ acts properly discontinuously on $\mathcal{M}$, is a good branchfold. In fact, there exists a PL $(\mathcal{G},\mathcal{M})$-covering $p : X \to O_X$ onto the good $(\mathcal{G},\mathcal{M})$-orbifold $O_X = \mathcal{M}/\mathcal{H}_X$, such that diagram 16 commutes.*

Proof. The existence of the map $p : X \to O_X$ follows from the fact that C_X is R_X-equivariant, being a completion of the R_X-equivariant map D_R. In order to prove that p is a $(\mathcal{G},\mathcal{M})$-map, let C_x be the local model of X at any point

$$\begin{array}{ccccc} & & \mathcal{M} & & \\ & {\scriptstyle C_X}\nearrow & & \searrow{\scriptstyle \pi_{\mathcal{H}_X}} & \\ \widetilde{X} & & & & O_X \\ & {\scriptstyle u_X}\searrow & & \nearrow{\scriptstyle p} & \\ & & X & & \end{array} \tag{16}$$

$x \in X$ and $(U_{\widetilde{x}}, P_x, V_x, \overline{G}_x = \overline{H}_x K_x)$ be the conical chart of $\widetilde{X}$ at any point $\widetilde{x} \in \widetilde{X}$ such that $u_X(\widetilde{x}) = x$. We consider the open subsets $C_X(U_{\widetilde{x}}) = g(V_x) \subset \mathcal{M}$ and $p(U_x) = \pi_{\mathcal{H}}(g(V_x)) \subset O_X$, where g is a suitable element of $\mathcal{G}$. By Proposition 1.15, the restriction $\pi_{\mathcal{H}|} : g(V_x) \to p(U_x)$ is a regular branched covering induced by the action of a finite subgroup $L \leqslant \mathcal{H}$, and we can identify it with the canonical projection π_L. Thus we have the commutative diagram below, saying that p is $(\mathcal{G}, \mathcal{M})$-map at x.

$$\begin{CD} V_x @>{g}>> g(V_x) \\ @A{p_{K_x}}AA @AA{\mathrm{id}}A \\ P_x @>{g \circ p_{K_x}}>> g(V_x) \\ @V{p_{H_x}}VV @VV{\pi_L}V \\ U_x @>{p_|}>> p(U_x) \end{CD} \tag{17}$$

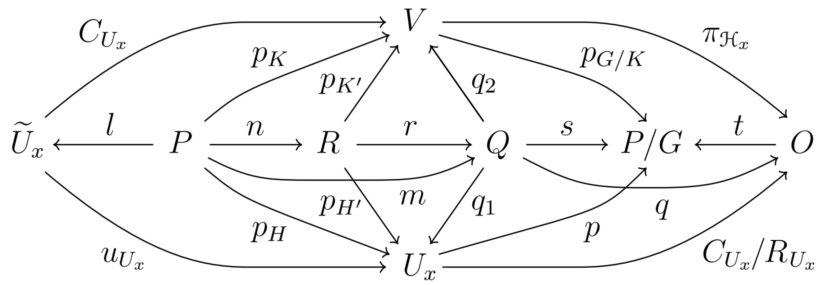
By the compactness of X, p is a PL map, hence a PL $(\mathcal{G}, \mathcal{M})$-covering. □

As we already observed, all branchfolds are locally very good. However, the construction of diagram 16, once adapted to the non-compact (and even non-complete) context of branchfold charts, gives us more information in the case of $(\mathcal{G}, \mathcal{M})$-branchfold. In particular, it allows us to see how the local holonomy group of a $(\mathcal{G}, \mathcal{M})$-branchfold at a point is related to the conical $(\mathcal{G}, \mathcal{M})$-charts centered at that point. This relation is stated by the following proposition.

PROPOSITION 3.13. *For any conical $(\mathcal{G}, \mathcal{M})$-chart $(U_x, P, V, G = HK)$ centered at x, the local holonomy group $\mathcal{H}_x = H_x(\pi_1(U_x - \Sigma U_x))$ coincides with $G/K \leqslant \mathcal{G}$.*

Proof. First of all, we observe that $\mathcal{H}_x \leqslant G/K$. In fact, the homomorphism $i_* : \pi_1(U_x - p^{-1}(\Sigma(P/G))) \to \pi_1(U_x - \Sigma U_x)$ induced by the inclusion is surjective and $H_x \circ i_* = H_{P/G} \circ p_{|*}$, where $p_| : U_x - p^{-1}(\Sigma(P/G)) \to P/G - \Sigma(P/G)$ is the restriction over $P/G - \Sigma(P/G)$ of the covering $p : U_x \to P/G$ associated to the chart. Hence, $\mathcal{H}_x = \operatorname{Im} H_x \leqslant \operatorname{Im} H_{P/G} = \mathcal{H}_{P/G} = G/K \leqslant \mathcal{G}$. To see that $\mathcal{H}_x = G/K$, look at the commutative diagram 18, representing the diagram of the $(\mathcal{G}, \mathcal{M})$-chart $(U_x, P, V, G = HK)$ and the maps of diagram 16. Since P is a simply connected pure branchfold covering of U_x, by Proposition 2.22 there exists a PL branchfold covering $l : P \to \widetilde{U}_x$ such that $u_{U_x} \circ l = p_H$. By a suitable choice of C_{U_x}, we can assume that $C_{U_x} \circ l = p_K$. Moreover, as $\mathcal{H}_x \leqslant G/K$ is a finite group, $\pi_{\mathcal{H}_x} : V \to O$ is a finite PL branchfold covering onto an orbifold O and there exists a PL branchfold covering $t : O \to P/G$. Similarly, C_{U_X}/R_{U_x} is a PL branchfold covering, as it is a finite map.

Now, let $q : Q \to O$ be the pullback of C_{U_x}/R_{U_x} and $\pi_{\mathcal{H}_x}$, with the associated PL branchfold coverings q_1 and q_2, and $q \circ r : R \to O$ be the minimal regularization of q.



(18)

Then, we define the coverings $p_{H'} = q_1 \circ r$ and $p_{K'} = q_2 \circ r$, with deck transformations groups H' and K' respectively. Denoting by L the group of deck transformations of $q \circ r$, we have $H' \leqslant L$ and $K' \trianglelefteq L$. Hence, we can consider the subgroup $G' = H'K' \leqslant L$ and the branchfold chart $(U_x, R, V, G' = H'K')$. By the universal property of the pullback, there exists a PL branchfold covering $m : P \to Q$ which commutes with the other maps. Furthermore, $q \circ m$ is regular, since $t \circ q \circ m = \pi_G$ is. Hence, by the universal property of the minimal regularizations, there exists a factorization $m = r \circ n$, for a PL branchfold covering $n : P \to R$. This gives a domination of the chart $(U_x, P, V, G = HK)$ on the chart $(U_x, R, V, G' = H'K')$. As a consequence, the last is a $(\mathcal{G}, \mathcal{M})$-chart and $\pi_G = \pi_{G'} \circ n : P \to P/G \cong R/G'$. On the other hand, since $G' \leqslant L$ there is a PL branchfold covering $P/G \cong R/G' \to O$, which commutes with the other maps. Then, we can conclude that $G/K \leqslant \mathcal{H}_x$. $\square$

Rational conifolds as geometric branchfolds

In this subsection we focus on the geometric branchfolds modelled on constant curvature Riemannian geometries. In particular, we apply the geometric goodness theorem to establish the relation between such branchfolds and conifold spaces.

We denote by $\mathcal{M}_k^m$ the m-dimensional model Riemannian space of constant curvature k and by $\mathcal{G}_k^m$ its isometry group, for any real number k. Hence, the geometric branchfolds we want to consider are the $(\mathcal{M}_k^m, \mathcal{G}_k^m)$-branchfolds.

The definition of *conifold of dimension m and curvature k*, in short *(m,k)-conifold*, is given by induction on the dimension m. The $(1,k)$-conifolds are the circles of any length, independently of k. For $m \geq 2$, an (m,k)-conifold X is a complete metric space locally modelled on k-cones over $(m-1,1)$-conifolds. More precisely, for any $x \in X$ there exist $\varepsilon > 0$ and an isometry between the open ball $B(x,\varepsilon)$ and the open k-cone $C_{k,\varepsilon}(L_x)$ on a connected compact $(m-1,1)$-conifold L_x, letting x correspond to the apex of the cone. Here, by the open *k-cone* $C_{k,r}(L)$ of a metric space L, with $r \leq k/\sqrt{\pi}$ if $k > 0$, we mean the open cone $L \times [0, r[\,/L \times \{0\}$ endowed with the metric $d((x_1,t_1),(x_2,t_2)) = d_{\mathcal{M}_k^2}(p_1,p_2)$, for a geodesic triangle p_0, p_1, p_2 in $\mathcal{M}_k^2$, such that $\angle_{p_0} = \min(d_L(x_1,x_2),\pi)$, $d(p_0,p_1) = t_1$ and $d(p_0,p_2) = t_2$.

Any $(\mathcal{G}_k^m, \mathcal{M}_k^m)$-branchfold X can be endowed with a natural metric, whose restriction to $X - \Sigma X$ has constant curvature k. Our first aim is to prove that such a metric turns X into a (m,k)-conifold. We define the *natural metric* on a $(\mathcal{G}_k^m, \mathcal{M}_k^m)$-branchfold X in the following way. First we lift the metric of $\mathcal{M}_k^m$ to R through the developing map $D_R : R \to \mathcal{M}$. By the R_X-invariance of D_R, the deck transformations of the ordinary covering $r = u_{X|} : R \to X - \Sigma X$ preserve such metric. Hence we have an induced metric on $X - \Sigma X$ turning r into a local isometry. Since $\Sigma X \subset X$ is a good subpolyhedron, we can extend by continuity the geodesic distance on $X - \Sigma X$

to a distance on X. Actually, in the light of Theorem 3.12, this natural metric on X can be obtained by putting the quotient metric on $O_X = \mathcal{M}/\mathcal{H}_X$ and then lifting this metric to X through the branched covering $p : X \to O_X$.

The above constructions of the natural metric on X can be also performed locally at $x \in X$. More precisely, for a local $(\mathcal{G}_k^m, \mathcal{M}_k^m)$-model C_x, we can either lift the metric of V_x to P_x through p_{K_x} and then consider the quotient metric on $U_x = P_x/H_x$, or lift to U_x through p_x the metric on $P_x/G_x \cong V_x/(G_x/K_x)$ induced by the last quotient. In both cases, if U_x is a sufficiently small convex conical neighborhood of x, we get the restriction to U_x of the natural metric of X.

Now, let us consider the linearization of C_x at x. This is the local model $(T_xU_x, T_{\widetilde{x}}P_x, T_{\overline{x}}V_x, G_x = H_xK_x)$, where T denotes the tangent cone, $\widetilde{x}$ and $\overline{x}$ are respectively the apices of P_x and V_x, while the action of G_x on $T_{\widetilde{x}}P_x$ is the unique which preserves the radial structure and corresponds to the original one on P_x in a neighborhood of $\widetilde{x}$ through the exponential map. Property (1) in Definition 3.1 is essential for the existence of such an action, since $\mathcal{G}_k^m$-admissible subpolyhedra of $\mathcal{M}_k^m$ stratify by totally geodesic submanifolds. In particular, we identify $(T_{\overline{x}}V_x, \overline{x})$ with $(R^m, 0)$ in such a way that under this identification G_x/K_x acts on it as a subgroup of $\mathrm{SO}(m)$. Then, we define $T^1_{\widetilde{x}}P_x = (T_{\widetilde{x}}p_{K_x})^{-1}(S^{m-1}) \subset T_{\widetilde{x}}P_x$, $T^1_{\widehat{x}}(P_x/G_x) = T_{\overline{x}}p_{G_x/K_x}(S^{m-1}) \subset T_{\widehat{x}}(P_x/G_x)$ and $L_xX = T_{\widetilde{x}}p_{H_x}(T^1_{\widetilde{x}}P_x) = (T_xp)^{-1}(T^1_{\widehat{x}}(P_x/G_x))$. Clearly, $T^1_{\widehat{x}}(P_x/G_x) = S^{m-1}/(G_x/K_x)$ is a very good $(\mathcal{G}_1^{m-1}, \mathcal{M}_1^{m-1})$-orbifold, hence L_xX is a very good $(\mathcal{G}_1^{m-1}, \mathcal{M}_1^{m-1})$-branchfold.

PROPOSITION 3.14. *Any $(\mathcal{G}_k^m, \mathcal{M}_k^m)$-branchfold X with its natural metric is a (m, k)-conifold. In fact, for every point $x \in X$ there exists $\varepsilon > 0$ such that the open geodesic ball $B(x, \varepsilon)$ is isometric to the k-cone $C_{k,\varepsilon}(L_xX)$, by an isometry letting x correspond to the apex of the cone.*

Proof. We proceed by induction on m. If $m = 1$ there is nothing to prove. We assume $m > 1$ and consider any point $x \in X$. Since the restriction of the natural metric of X to L_xX coincides with the natural metric of L_xX as a $(\mathcal{G}_1^{m-1}, \mathcal{M}_1^{m-1})$-branchfold, by induction L_xX is a $(m-1, 1)$-conifold. We look at the diagram below.

$$\begin{array}{ccc} C_{k,\varepsilon}(S^{m-1}) & \longrightarrow & B(\overline{x}, \varepsilon) \subset V_x \\ \uparrow & & \uparrow \\ C_{k,\varepsilon}(T^1_{\widetilde{x}}P_x) & \longrightarrow & B(\widetilde{x}, \varepsilon) \subset P_x \\ \downarrow & & \downarrow \\ C_{k,\varepsilon}(L_xX) & \longrightarrow & B(x, \varepsilon) \subset U_x \end{array} \tag{19}$$

The vertical arrows on the right are restrictions of the coverings p_{H_x} and p_{K_x} associated to the $(\mathcal{G}_k^m, \mathcal{M}_k^m)$-chart C_x, while those on the left are obtained by applying the cone construction $C_{k,\varepsilon}$ to restrictions of the corresponding maps associated to the linearization of such $(\mathcal{G}_k^m, \mathcal{M}_k^m)$-chart. The horizontal arrows are induced by the exponential map on the top. As this is an isometry and the vertical arrows are local isometries out of the singularities, the other horizontal maps are isometries too. □

In order to characterize the conifolds that can be obtained from branchfolds as in the above proposition, we need the notion of (local) holonomy of a conifold. Any

(m, k)-conifold X is a pseudo-manifold of dimension m. Moreover, the *singular locus* $\Sigma X = \{x \in X \mid X \text{ is not an } m\text{-manifold at } x\}$ has dimension $\leq m - 2$, hence it is a good subpolyhedron of X. On the other hand, the complement $X - \Sigma X$ is a $(\mathcal{G}^m_k, \mathcal{M}^m_k)$-manifold. Then, the holonomy $H_X : \pi_1(X - \Sigma X) \to \mathcal{G}^m_k$ is defined and we call it the *holonomy* of X. Similarly, the holonomy $H_x : \pi_1(B(x, \varepsilon) - \Sigma B(x, \varepsilon)) \to \mathcal{G}^m_k$ is defined for $\varepsilon > 0$ sufficiently small and we call it the *local holonomy* of X at x.

By a *rational conifold* we mean a conifold X such that the local holonomy group $\mathcal{H}_x = \operatorname{Im} H_x \leqslant \mathcal{G}^m_k$ is finite for every $x \in X$. We chosed this terminology because the local holonomy at a codimension 2 point $x \in \Sigma X$ is finite if and only if the singular angle of X at x is a rational multiple of π radians.

Theorem 3.15. *A connected (m, k)-conifold X admits a $(\mathcal{G}^m_k, \mathcal{M}^m_k)$-branchfold structure as in Proposition 3.14 if and only if it is a rational conifold.*

Proof. The "only if" part is quite trivial. In fact, the local holonomies of X as a conifold and as a branchfold are the same. Then, the finiteness of $\mathcal{H}_x$, for every $x \in X$, derives from Proposition 3.13.

We prove the "if" part by induction on m, the case $m = 1$ being trivial. Let us assume that X is a connected (m, k)-conifold with $m > 1$. Given any $x \in X$, there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \cong C_{k,\varepsilon}(L_x)$ for some connected compact $(m - 1, 1)$-conifold L_x. Moreover, the holonomy group $\mathcal{H}_{L_x}$ of L_x is finite, since it coincides with the local holonomy group $\mathcal{H}_x$ of X at x. By the inductive hypothesis, L_x is a $(\mathcal{G}^{m-1}_1, \mathcal{M}^{m-1}_1)$-branchfold. Hence it is a good branchfold. Actually, L_x is very good, being a finite branchfold covering of the very good orbifold $\mathcal{G}^{m-1}_1 / \mathcal{H}_x = S^{m-1} / \mathcal{H}_x$. This allows us to construct the following diagram of finite branchfold coverings.

$$
\begin{array}{ccccc}
 & & S^{m-1} & & \\
 & \stackrel{\pi_K}{\nearrow} & & \stackrel{\pi_{\mathcal{H}_x}}{\searrow} & \\
P & & & & O \\
 & \underset{\pi_H}{\searrow} & & \underset{p}{\nearrow} & \\
 & & L_x & &
\end{array}
\tag{20}
$$

By applying the k-cone construction $C_{k,\varepsilon}$ to all spaces and maps in the diagram, we get a conical $(\mathcal{G}^m_k, \mathcal{M}^m_k)$-chart $(U_x, P_x, V_x, G = HK)$ for X at x, where $U_x = B(x, \varepsilon) \cong C_{k,\varepsilon}(L_x)$, $P_x = C_{k,\varepsilon}(P)$, $V_x = C_{k,\varepsilon}(S^{m-1}) \cong B(0, \varepsilon) \subset \mathcal{M}^m_k$ and $P_x/G \cong C_{k,\varepsilon}(O)$ (cf. Proposition 3.13). In order to prove that the conical $(\mathcal{G}^m_k, \mathcal{M}^m_k)$-charts we have just constructed form a $(\mathcal{G}^m_k, \mathcal{M}^m_k)$-atlas, it suffices to verify that for any two such charts $(U_x, P_x, V_x, G = HK)$ and $(U_y, P_y, V_y, \overline{G} = \overline{H}\,\overline{K})$ with $U_x \subset U_y$, we have that the former is equivalent to a restriction $(U_x, P'_x, V'_x, G' = H'K')$ of the latter to U_x. Since the above construction of $(\mathcal{G}^m_k, \mathcal{M}^m_k)$-charts for X originates from developing maps (cf. diagrams 15 and 16), we can assume, up to multiplication by elements of $\mathcal{G}$, that $V_x = V'_x \subset V_y$ and $G/K = \mathcal{H}_{U_x} \leqslant \mathcal{H}_{U_y} = \overline{G}/\overline{K}$, where the equalities follows from Proposition 3.13. On the other hand, any restriction of a $(\mathcal{G}^m_k, \mathcal{M}^m_k)$-chart is still a $(\mathcal{G}^m_k, \mathcal{M}^m_k)$-chart, so by Proposition 3.13 $G'/K' = \mathcal{H}_{U_x}$. Then, $P_x/G = P'_x/G'$ and we have the same PL branchfold covering $p = C_{U_x}/R_{U_x} : U_x \to P_x/G = P'_x/G'$ associated to the charts we are comparing. Therefore, those charts induce the same branchfold structure on U_x, hence they are equivalent. □

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